

# Riccati equations and optimal control of well-posed linear systems

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**Abstract:** We generalize the classical theory on algebraic Riccati equations and optimization to infinite-dimensional well-posed linear systems, thus completing the work of George Weiss, Olof Staffans and others. We show that the optimal control is given by the stabilizing solution of an integral Riccati equation. If the input operator is not maximally unbounded, then this integral Riccati equation is equivalent to the algebraic Riccati equation.

Using the integral Riccati equation, we show that for (nonsingular) minimization problems the optimal state-feedback loop is always well-posed. In particular, the optimal state-feedback operator is admissible also for the original semigroup, not only for the closed-loop semigroup (as has been known in some cases); moreover, both settings are well-posed with respect to an external input. This leads to the positive solution of several central, previously open questions on exponential, output and dynamic (aka. “internal”) stabilization and on coprime factorization of transfer functions.

Our theory covers all quadratic (possibly indefinite) cost functions, but the optimal state feedback need not be well-posed (admissible) unless the cost function is uniformly positive or the system is sufficiently regular.

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**Keywords:** Regular linear system, integral Riccati equation, algebraic Riccati equation, stabilizing solution, optimal state feedback, exponential stabilization, dynamic stabilization, internal stabilization, internal loop, optimizability, finite cost condition, quasi-right coprime factorization, doubly coprime factorization, Popov function.

## 1 Introduction: systems with bounded generators

In this section we present a (mostly known) very special case of our results. At the end of this section and in “Conclusions” (Section 13) we explain, how we have generalized these results to more general systems, cost functions and stability goals, in the other sections.<sup>1</sup>

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<sup>1</sup>This is the March 14, 2004 draft (= the latest version before its split) as such except for this publication footnote (February 27, 2016). I was asked to publish it now in arXiv to allow referencing to results not published elsewhere. In an earlier form it was circulated a few of months earlier. Later, parts of it were published, usually with several newer results: “State-Feedback Stabilization of Well-Posed Linear Systems” *Integral Equations and Operator Theory* 55 (2), pp. 249-271, 2006 (early/middle parts). “Coprime factorization and dynamic stabilization of transfer functions”, *SIAM Journal on Control and Optimization*, 45 (6), pp. 1988-2010, 2007 (not systems, just transfer functions, unlike in the ones mentioned below). “Weakly coprime factorization and state-feedback stabilization of discrete-time systems” *Mathematics of Control, Signals, and Systems*, 20 (4), pp. 321-350, 2008, “Weakly coprime factorization and continuous-time systems” *IMA Journal of Mathematical Control and Information*, 25 (4): pp. 515-546, 2008. doi:10.1093/imamci/dnn011 Many of the results were in some form already in [M02]. Most remaining main results, such as Theorem 5.21 and output and measurement feedback stabilization results for WPLSs were published in “Coprime factorizations and stabilization of infinite-dimensional linear systems” *Proceedings of CDC-ECC2005*. Of those results I had two corresponding drafts fairly ready late 2007 but then had to stop finishing them due to other responsibilities. I will probably publish also them in arXiv as such, if I do not find time to update their references and shorten the presentation.

In the most simple case, a linear time-invariant control system is governed by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0 \end{aligned} \quad (1)$$

(for  $t \geq 0$ ), where the *generators*  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(H \times U, H \times Y)$  are matrices, or more generally, linear operators on Hilbert spaces  $(U, H, Y)$  of arbitrary dimensions. There  $u$  is the *input* (or *control*),  $x$  the *state* and  $y$  the *output* of the system. Obviously,  $x_0$  and  $u$  determine  $x$  and  $y$  uniquely. In this section, we shall allow  $A$  to be unbounded as long as it generates a strongly continuous semigroup, which we denote by  $e^{At}$ ; in later sections also  $B$  and  $C$  may be unbounded.

By  $\mathcal{B}(H, U)$  we denote the space of bounded linear operators  $H \rightarrow U$ , by  $\mathbb{R}_+$  the set  $[0, \infty)$  and by  $L^2(\mathbb{R}_+; U)$  the Banach space of (equivalence classes of Bochner) measurable functions  $u : \mathbb{R}_+ \rightarrow U$  for which  $\|u\|_2^2 := \int_0^\infty \|u(t)\|_U^2 dt < \infty$ .

We first take a look at the following (LQR) minimization problem. Given any *initial state*  $x_0 \in H$ , we want to minimize a cost function, such as

$$\mathcal{J}(x_0, u) = \int_0^\infty (\|x(t)\|_H^2 + \|u(t)\|_U^2) dt. \quad (2)$$

Observe that the output  $y$  (and hence  $C$  and  $D$  too) is irrelevant to this problem.

A necessary condition for the existence of a minimum is the *state-FCC* (Finite Cost Condition):

For each  $x_0 \in H$ , there exists some control  $u \in L^2(\mathbb{R}_+; U)$  such that  $x \in L^2(\mathbb{R}_+; H)$  (3)

(i.e.  $\inf_u \mathcal{J}(x_0, u) < \infty$  for all  $x_0$ , so that we do not have to optimize over the empty set). Thus, some *stable* input ( $u \in L^2$ ) must make the state stable ( $x \in L^2$ ). It is known that the state-FCC is also sufficient:

**Theorem 1.1** ( $\int_0^\infty \|x\|^2 + \|u\|^2$ ) *The following are equivalent:*

- (i) *For each initial state  $x_0 \in H$ , there exists a unique control that minimizes (2).*
- (ii) *The algebraic Riccati equation (ARE)*

$$\mathcal{P}B B^* \mathcal{P} = A^* \mathcal{P} + \mathcal{P}A + I \quad \text{on } \text{Dom}(A) \quad (4)$$

*has a solution  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$  that is exponentially stabilizing, i.e., such that the  $C_0$ -semigroup  $e^{t(A+BK)}$  is exponentially stable,<sup>2</sup> where  $K := -B^* \mathcal{P}$ .*

- (iii) *The state-FCC (3) holds.*

*Assume that (ii) has a solution. Then this solution is unique, and the (state-feedback) control  $u(t) = Kx(t)$  strictly minimizes the cost (2) for any initial state  $x_0 \in H$ . Moreover, the minimal cost equals  $\langle x_0, \mathcal{P}x_0 \rangle_H$ .* □

This is a special case of Corollary 6.6(a). In fact, a solution of (4) is exponentially stabilizing iff it is nonnegative.

By the *transfer function* of the system (1) we mean the map  $s \mapsto \hat{\mathcal{D}}(s) \in \mathcal{B}(U, Y)$ , where

$$\hat{\mathcal{D}}(s) := D + C(s - A)^{-1}B. \quad (5)$$

When  $x_0 = 0$ , we have  $\hat{y}(s) = \hat{\mathcal{D}}(s)\hat{u}(s)$  for each  $s$  on some right half-plane; here  $\hat{u}(s) := \int_0^\infty e^{-st}u(t) dt$  is the *Laplace transform* of  $u$ . This fact follows from the identity

$$(s - A)\hat{x}(s) = x_0 + B\hat{u}(s), \quad (6)$$

which is a direct consequence of (1) (and Lemma B.2).

If we allow for an *external input*  $u_\zeta \in L^2(\mathbb{R}_+; U)$  to the state-feedback loop of Theorem 1.1, i.e.  $u(t) = Kx(t) + u_\zeta(t) \forall t \geq 0$ . For  $x_0 = 0$  this leads to  $(s - A)\hat{x}(s) = B(K\hat{x}(s) + \hat{u}_\zeta(s))$ , i.e., to

$$\hat{x}(s) = (s - A - BK)^{-1}Bu_\zeta(s), \quad \hat{u} = \mathcal{M}\hat{u}_\zeta, \quad \hat{y} = \mathcal{N}\hat{u}_\zeta, \quad (7)$$

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<sup>2</sup> $\|e^{t(A+BK)}\| \leq Me^{-\epsilon t}$  for some  $M, \epsilon > 0$  and all  $t > 0$  (cf. Lemma 2.2).

on some right half-plane, where  $\hat{\mathcal{M}}(s) := I + K(s - A - BK)^{-1}B$ ,  $\hat{\mathcal{N}}(s) = D + (C + DK)(s - A - BK)^{-1}B$ .

We call a state-feedback operator  $K : \text{Dom}(A) \rightarrow U$  *admissible* for the system (1) if the map  $u_\zeta \rightarrow u$  and its inverse are locally bounded in  $L^2$ . An equivalent requirement is that  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{M}}^{-1}$  are bounded on some right half-plane. A sufficient condition is that  $K$  is *bounded* ( $K \in \mathcal{B}(H, U)$ ), but in a more general setting with an unbounded  $B$  ( $B \notin \mathcal{B}(U, H)$ ) one sometimes needs an unbounded  $K$  to make  $e^{(A+BK)t}$  stable.

Theorem 1.1 implies the following:

**Corollary 1.2** *The system satisfies the state-FCC (3) iff it is exponentially stabilizable.*

*Exponentially stabilizable* means that there exists an admissible  $K$  s.t. the semigroup generated by  $A + BK$  is exponentially stable. Our generalization of Corollary 1.2 (Corollary 5.2) solves positively the “optimizability = exponential stabilizability” problem studied in, e.g., [WR00].

Similar results also hold for the alternative (LQR) cost function

$$\mathcal{J}(x_0, u) = \int_0^\infty (\|y(t)\|_Y^2 + \|u(t)\|_U^2) dt : \quad (8)$$

**Theorem 1.3** ( $\int_0^\infty \|y\|^2 + \|u\|^2$ ) *Assume that  $D = 0$ . Then the following are equivalent:*

- (i) *For each initial state  $x_0 \in H$ , there exists a unique control that minimizes (8).*
- (ii) (ARE) *The algebraic Riccati equation*

$$\mathcal{P}BB^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + C^*C \quad (9)$$

*has a nonnegative solution  $\mathcal{P} \in \mathcal{B}(H)$ .*

- (iii) (output-FCC) *For each  $x_0 \in H$ , there is  $u \in L^2(\mathbb{R}_+; U)$  s.t.  $y \in L^2(\mathbb{R}_+; Y)$ .*

*Assume that (ii) has a solution. Then there is a smallest nonnegative solution  $\mathcal{P} \in \mathcal{B}(H)$  of (9), and the (state-feedback) control  $u(t) = Kx(t)$  strictly minimizes the cost (8) for any initial state  $x_0 \in H$ , where  $K := -B^*\mathcal{P}$ . Moreover, the minimal cost equals  $\langle x_0, \mathcal{P}x_0 \rangle_H$ .*  $\square$

This is a special case of Corollary 6.6(b).

**Corollary 1.4** *The system satisfies the output-FCC 1.3(iii) iff it is output-stabilizable.*

*Output-stabilizable* means that there exists an admissible (state-feedback operator)  $K$  s.t.  $u, y \in L^2$  for each initial state  $x_0 \in H$  under  $u(t) = Kx(t)$ . Even more is true:  $u, y \in L^2$  for any  $x_0 \in H$  and  $u_\zeta \in L^2(\mathbb{R}_+; U)$ , and the maps  $u_\zeta \rightarrow \begin{bmatrix} y \\ u \end{bmatrix}$  are coprime in a sense which we will describe below if we choose  $K$  as in Theorem 1.3.

The above claim “ $u_\zeta \in L^2 \Rightarrow u, y \in L^2$ ” implies that the transfer functions  $\begin{bmatrix} \hat{\mathcal{N}} \\ \hat{\mathcal{M}} \end{bmatrix} : \widehat{u}_\zeta \rightarrow \begin{bmatrix} \hat{y} \\ \hat{u} \end{bmatrix}$  (have holomorphic extensions that) are bounded on the right half-plane  $\mathbb{C}^+ := \{s \in \mathbb{C} \mid \text{Re } s > 0\}$ . We also show that the maps  $\hat{\mathcal{N}}$  and  $\hat{\mathcal{M}}$  are *q.r.c.* (quasi-right coprime), which means that  $\begin{bmatrix} \hat{\mathcal{N}} \\ \hat{\mathcal{M}} \end{bmatrix} \hat{f} \in \widehat{L}^2 \Rightarrow f \in L^2 \forall f$  (see Definition 5.4(a)); this implies that  $\hat{\mathcal{N}}$  and  $\hat{\mathcal{M}}$  do not have common zeros on  $\mathbb{C}^+$  and is as good as the “standard right coprimeness” in typical applications (and equivalent to it at least if  $\dim U < \infty$  and  $\hat{\mathcal{N}}, \hat{\mathcal{M}}$  are continuous on  $\overline{\mathbb{C}^+} \cup \{\infty\}$ , by the proof of Lemma 5.12).

Using our generalization of Corollary 1.4, we show that any holomorphic map having a “stable (right) factorization” has a “q.r.c. factorization”:

**Theorem 1.5 (Right-coprime factorization)** *Given any holomorphic, bounded maps  $\hat{\mathcal{N}} : \mathbb{C}^+ \rightarrow \mathcal{B}(U, Y)$ ,  $\hat{\mathcal{M}} : \mathbb{C}^+ \rightarrow \mathcal{B}(U)$  such that  $\hat{\mathcal{M}}^{-1}$  exists and is bounded on some right half-plane, there are  $\hat{\mathcal{N}}_2, \hat{\mathcal{M}}_2$  that satisfy the same conditions,  $\hat{\mathcal{N}}\hat{\mathcal{M}}^{-1} = \hat{\mathcal{N}}_2\hat{\mathcal{M}}_2^{-1}$ , and, in addition,  $\hat{\mathcal{N}}_2$  and  $\hat{\mathcal{M}}_2$  are q.r.c.*

Thus, we can “cancel any common zeros of  $\hat{\mathcal{N}}$  and  $\hat{\mathcal{M}}$  on  $\mathbb{C}^+$ ”. This and further equivalent conditions on the map  $\hat{\mathcal{N}}\hat{\mathcal{M}}^{-1}$  are given in Corollary 5.13.

By applying Theorem 1.1 to the *dual system*  $\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$  in place of  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ , we see that the “dual” of the state-FCC (3) holds iff there exists  $H \in \mathcal{B}(Y, H)$  s.t.  $A + HC$  generates an exponentially stable semigroup. This and (3) lead to so called *doubly coprime factorization (d.c.f.)* of the transfer function  $\hat{\mathcal{D}}$  and to *dynamic (output-feedback) stabilization* of the system. Conversely, dynamic stabilization leads to a d.c.f., by Lemma 5.20 below; this is an infinite-dimensional version of the result [S89] by Malcolm Smith. See Corollary 5.7 and Theorem 5.17 and the references below them for details. Note that dynamic (I/O-)stabilization is the same as “internal stabilization” in, e.g., [Q03], except that we require the I/O map and the controller to be well-posed, i.e., both transfer functions must be bounded on some right half-plane.

The above results are well-known for bounded  $B$  (the same applies to most results mentioned in the remainder of this section if we ignore the IREs), except for the claims on coprimeness, which have been known for finite-dimensional  $U, H, Y$  only. In this article, we shall generalize the above results to WPLSs (see below) and to general quadratic cost functions in place of  $\mathcal{J}$  (including those that are indefinite with respect to  $u$ ). Also some other results are presented. However, if  $B$  and  $C$  are extremely unbounded, then one must use integral Riccati equations instead of the algebraic ones above and, in the case of indefinite  $\mathcal{J}$ , the optimal state-feedback need no longer be admissible.

In Section 2, we shall define *WPLSs* (well-posed linear systems, or the Salamon–Weiss class), which form a generalization of (1) allowing for rather unbounded  $B$  and  $C$  (the “feedthrough” operator  $D = \lim_{s \rightarrow +\infty} \hat{\mathcal{D}}(s)$  need not exist; if it does, then the WPLS is called *regular*).

In Section 3, we recall what *state feedback* (the above equation  $u = Kx$ ) is in the WPLS context.

In Section 4, we shall define a general domain of optimization (“the set of admissible inputs  $u$  for a given initial state  $x_0$ ”) to replace its special cases (the set of  $u$ ’s in (3) or those in Theorem 1.3(iii)). Then we define a general cost function  $\mathcal{J}$  and give sufficient conditions for the existence of an optimal control, i.e., a control that makes the derivative of the cost function vanish. If the cost function is nonnegative, then such a control is cost-minimizing; in the general case it corresponds to a saddle point (“maximin”) control, which is used to solve, e.g.,  $H^\infty$  control problems (“the best control for the worst disturbance”; see [M02]).

In Section 5, we show that for uniformly positive quadratic cost functions (such as (2) and (8)), under the (generalized) FCC, there is always a unique cost-minimizing state feedback. The existence of a unique optimal control has been well known (see, e.g., [FLT88] or [Z96] for the cost function (8)), but it has not been known that it is given by (well-posed) state feedback. The corollaries of this result, also given in Section 5, are perhaps the main results of this article — most of results 1.1–1.5 are special cases of some of them.

In Section 6 we shall generalize Theorems 1.1 and 1.3: we shall show that for any regular WPLS and any quadratic (possibly indefinite) cost function (and any typical domain of optimization), there is a (regular) optimal state-feedback operator ( $K$ ) iff the ARE has a stabilizing solution. Here *stabilizing* means that the resulting controlled system is stable in the sense corresponding to the domain of optimization (cf. Theorem 1.1(iii)). The necessity of the ARE was originally discovered independently by Olof Staffans [S97] and Martin Weiss and George Weiss [WW97], for stable regular WPLSs. The author established the converse in [M97] and extended the equivalence to the unstable case in [M02]. This equivalence (Theorem 6.2) can be simplified in certain special cases, as we show in Sections 6 and 8 in [M02].

In Section 7, we generalize Theorem 6.2 to general WPLSs. Since the ARE cannot be defined for irregular systems, we use the integral Riccati equation (IRE) instead: the IRE has a stabilizing solution iff there is some (well-posed) optimal state-feedback for the WPLS (Theorem 7.2). (The ARE can be used only when  $\hat{\mathcal{D}}(+\infty)$  and  $\hat{\mathcal{M}}(+\infty)$  exist.)

In Corollary 7.5 we explain the results of Section 5 in terms of IREs and AREs and show that the word “stabilizing” can be replaced by “nonnegative” for the cost functions (2) and (8).

However, even if there exists a unique optimal control for each initial state, the optimal

control need not be given by any (well-posed) state feedback (except for uniformly positive cost functions, as shown in Section 5). To treat this most general case, we show that a unique optimal control is always given by a “generalized state feedback” (in the uniformly positive case this was already known [Z96]), and that a third equivalent condition is that a variant of the IRE has a stabilizing solution (Theorem 7.1).

Fortunately, if the original system is sufficiently regular (we give various alternative assumptions), then the “generalized state feedback” is nevertheless given by a well-posed, even regular state-feedback operator, thus making also the AREs and IREs equivalent to the three conditions mentioned above; this is explained in Sections 8 and 6. Section 8 focuses on systems for which  $e^{-A}B$  and  $Ce^{-A}B$  are locally integrable.

Most of Sections 9–12 consist of the proofs of the results mentioned above. Only the simplest proofs have been included in the previous sections.

Theorem 1.3 was “generalized” to WPLSs having a bounded output operator ( $C$ ) by Franco Flandoli, Irena Lasiecka and Roberto Triggiani in [FLT88], using an “ARE” given on  $\text{Dom}(A + BK)$  (although the well-posedness of  $K$  was not known before this article). We extend their result to regular WPLSs and to general cost functions and domains of optimization in Theorem 9.9; see Theorem 9.1 (and 4.7) for the irregular case. Also the other variants of the IRE are treated in Sections 9 and 10.

In Section 11 we study the coercivity of the cost function, which is a sufficient (and in many cases also necessary) condition for the existence of a unique optimal control.

In Section 13 (“Conclusions”), we summarize the Riccati equation and optimization theory of this article. The appendices contain some auxiliary results used in the proofs.

Thus, we generalize and extend most of the theory in [FLT88], [Z96], [S97]–[S98b], [WW97], [M97] and much of that in [M02]. Further notes are given at the end of each of the remaining sections. Additional notes are given in [M02], which also provides numerous further results, details, explanations, examples, applications and references for much of the theory presented here, as well as the corresponding discrete-time results.

**Notes for Section 1:** Corollaries 1.2 and 1.4 and Theorem 1.5 will be extended to general WPLSs in Corollary 5.2, Theorem 5.9 and Corollary 5.13, respectively. Theorems 1.1 and 1.3 will be extended to general WPLSs in Corollary 5.2 and Theorem 5.9, respectively, combined with Corollary 7.5(a)&(c); see (b) and the remarks below the corollary for cases where an ARE can be used instead of the IRE (by Theorem 6.2, in those cases a state-feedback operator can be used instead of a state-feedback pair). Further discussion on different extensions (and on what kind of extensions are not true) is given in Section 13.

## Notation:

$\exists, \forall$ :	$\exists$ = “there exists”, $\forall$ = “for all”.
$*$ :	unknown/omitted element (e.g., “ $X = \begin{bmatrix} I & 0 \\ * & * \end{bmatrix}$ ”).
$A^{-*}$ :	$A^*$ = the (Hilbert space) adjoint of $A$ ; $A^{-*} := (A^{-1})^* = (A^*)^{-1}$ .
$\langle \cdot, \cdot \rangle$ :	Inner product (usually in $L^2$ over $\mathbb{R}$ ).
$U, W, H, Y$ :	Hilbert spaces of arbitrary dimensions; cf. p. 7.
$\mathcal{B}(U, Y)$ :	Bounded linear maps $U \rightarrow Y$ . $\mathcal{B}(U) := \mathcal{B}(U, U)$ .
$A \gg 0$ :	$A \geq \epsilon I$ for some $\epsilon > 0$ .
$\text{Dom}(A)$ :	The domain of the semigroup generator $A$ with the graph norm $(\ x\ _H^2 + \ Ax\ _H^2)^{1/2}$ . See Lemma 2.4 for details and for $\text{Dom}(A^*)^* = H_{-1} \supset H$ and $\text{Dom}(A)^* \supset H$ .
$\mathcal{G}$ :	The subset (often group) of <i>invertible</i> elements (e.g., $T \in \mathcal{GB}(X, Y)$ if $ST = I_X$ and $TS = I_Y$ for some $S \in \mathcal{B}(Y, X)$ ).
$I$ :	The identity operator.
$\mathbb{R}_\pm, \mathbb{N}, \mathbb{R}, \mathbb{C}$ :	$\mathbb{R}_\pm := \pm[0, \infty)$ , $\mathbb{N} := \{0, 1, 2, \dots\}$ , $\mathbb{R}$ := real, $\mathbb{C}$ := complex numbers.
$\mathbb{C}_\omega^+$ :	The right half-plane $\{z \in \mathbb{C} \mid \text{Re } z > \omega\}$ ; $\mathbb{C}^+ := \mathbb{C}_0^+$ .
w-lim:	The weak limit: $\text{w-lim}_{n \rightarrow \infty} D_n = D \Leftrightarrow \langle D_n x, y \rangle \rightarrow \langle D x, y \rangle \forall x, y$ .
$\square$ :	“End of proof”; or at the end of a theorem/result: “no formal proof follows, see the following text for a proof/reference”.

“If” := “if and only if”, “s.t.” := “such that”, “w.l.o.g.” := “without loss of generality”, “w.r.t.” := “with respect to”, “one-to-one” := “injective” (i.e.,  $f(x) = f(y) \Rightarrow x = y$ ).

We try to explain the rest of the notation as it appears, hence the reader may skip the rest of this section at this stage and use it later to find forgotten symbols or terms.

See the following pages (or formulas) for the following symbols:  $[\frac{A}{C}|\frac{B}{D}]$  8&9, (1),  $(\frac{A}{C}|\frac{B}{D})$  9,  $[\frac{A}{C}|\frac{B}{D}]$  7, (10);  $(A, B), (A | B), [\mathcal{A} | \mathcal{B}]$  17;  $H_1, H_{-1}$  8,  $H_{\text{strong}}^2$  40,  $H^\infty$  9,  $H_\infty^\infty$  8,  $H_B$  31,  $J, \mathcal{J}$  (32),  $[\mathcal{K} | \mathcal{F}]$ ,  $K, F$  11&12&36,  $L_\omega^2, L_\omega^1$  2&7,  $L_c^2 := \{u \in L^2 \mid u \text{ has a compact support}\}$ ;  $\mathcal{M}, \mathcal{N}$  11&36,  $\mathcal{P}$  15&34&31&40&36;  $\mathcal{R}f(t) := f(-t)$ ;  $\mathcal{S}_{\text{PT}}$  (Popov Toeplitz operator) 55,  $\mathcal{U}_{\text{exp}}$  (29),  $\mathcal{U}_{\text{out}}$  (30),  $\mathcal{U}_{\text{str}}$  13,  $\mathcal{U}_*$  13,  $\|\cdot\|_{\mathcal{U}_*}$  14&15,  $u, x, y, x_0$  8,  $W_\omega^{1,2}$  53,  $Z^s, Z^u, \dot{Y}, \mathcal{Q}, \mathcal{R}$  13,  $\pi_\pm$  7,  $\pi_E$  7,  $\rho(A) := \sigma(A)^c$ ,  $\rho_\infty(A)$  68,  $\tau$  7,  $\chi_E$  7,  $\omega_A$  7.

By  $A, B, C, D, K, F, M, N, X$  we denote the generators (pp. 2&8&9) of  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{K}, \mathcal{F}, \mathcal{M}, \mathcal{N}, \mathcal{X}$  respectively; similarly for other pairs of capital and script letters (and sub- and superscripts).

Subscripts:  $\hat{\mathcal{D}}_\Sigma$  68,  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}$  49;  $\Sigma_+$  55,  $\Sigma_\odot$  12&36,  $C_c, D_c$  68,  $\Sigma_{\text{ext}}$  11,  $\Sigma_L$  10,  $\Sigma_{\text{opt}}$  15;  $B_w^*$  31&9,  $C_w, K_w$  9.

Superscripts:  $\hat{\cdot}$  9&68,  $\check{\cdot}$  68,  $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{K}}, \hat{\mathcal{D}}, \hat{\mathcal{X}}, \hat{\mathcal{F}}, \hat{\mathcal{M}}$  8&68,  $B^*, C^*, K^*$  8;  $\Sigma^d$ : see “dual system” below;  $\Sigma^\tau$  7;  $\mathcal{A}^t$ ;  $\mathcal{B}^t := \mathcal{B}\tau^t\pi_+$ ;  $\mathcal{C}^t := \pi_{[0,t)}\mathcal{C}$ ,  $\mathcal{K}^t$ ;  $\mathcal{D}^t := \pi_{[0,t)}\mathcal{D}\pi_{[0,t)}$ ,  $\mathcal{F}^t$ ,  $\mathcal{X}^t$ ,  $\mathcal{M}^t$ ,  $\mathcal{N}^t$  (14). Non-generic symbols having superscripts:  $P^t := \pi_{[0,t)} + \tau^{-t}\mathcal{K}_0\mathcal{B}^t$  43,  $\mathcal{S}^t$  (43b) & 58,  $\hat{\mathcal{S}}$  (44b).

Acronyms: ARE:=Algebraic RE 31,  $B_w^*$ -ARE 33, FCC means that  $\mathcal{U}_*(x_0) \neq \emptyset \forall x_0 \in H$  (cf. pp. 15&2&3);  $\Sigma_{\text{opt}}$ -IRE 41,  $\widehat{\Sigma_{\text{opt}}}$ -IRE 41, IRE:=integral RE 36; q.r.c., r.c., d.c. 17&12&19; RE:=Riccati equation, MTIC 40, MTIC<sup>L1</sup> 38, RCC 31, rconn 68,  $\mathcal{S}^t$ -IRE,  $\hat{\mathcal{S}}$ -IRE 34, TIC := TIC<sub>0</sub>, TIC <sub>$\omega$</sub>  8, WPLS 7, WR, SR, UR, ULR 9.

Terms: *adjoint* see dual; *admissible* 11&31&36&10, *bounded*  $B, C$  9,  $B$  not maximally unbounded 32, *characteristic function* 68, *closed-loop system* 10&12, *control in WPLS form* 10, *coprime* 17, *cost function*  $\mathcal{J}$  14, *detectable* 17&19, *discrete subset* 69, *dual system* ( $\Sigma^d = (\frac{A^*}{C^*}|\frac{B^*}{D^*})$ ,  $\mathcal{D}^d(s) = \hat{\mathcal{D}}(\bar{s})^*$ ): p. 4 and [M02], *dynamic feedback controller* 19, *estimatable* 17, *exponentially stable* 2&7, *exponentially stabilizing* 2&12, *external input*  $u_\odot, u_L$  11&2&10, *factorization* 17&28, *feedthrough* 9, *generators* 2&8&9,  $J$ -coercive =  $\mathcal{S}_{\text{PT}} \in \mathcal{GB}$  = invertible Popov Toeplitz operator (= “no invariant zeros” =  $\|\mathcal{D}u\|_{\mathcal{U}_*} \geq \|u\|$  ( $u \in \mathcal{U}_*(0)$ ) if  $J \gg 0$ ) 15&56&55,  $J$ -optimal = “optimal” (= “minimizing” if  $J \geq 0$ ) 14&31, *jointly stabilizable and detectable* 19, *meromorphic* 21&69, *nondiscrete* 69, *optimizable* 16, *output-FCC* 3, *output-stabilizing* 7&3, *Popov* 15, *Pritchard–Salamon systems* 34, *realization* 9, *regular* 9, *Riccati operator*  $\mathcal{P}$  ( $J$ -optimal cost operator) 15, *signature* 35, *SOS-stable* 20&12&7, *stabilizing* 12, see also “ $\mathcal{U}_*$ -stabilizing”; *stabilizable* 12, *stable* 7&20&2&17, *state feedback* 11, *state-FCC* 2, *transfer function* 8&2&69,  $\mathcal{U}_*$ -stabilizing 31&36&41, *Yosida extension* 9.

Most of the notation and terminology and some proofs and further results are presented in greater detail in [M02] (under the replacements  $\mathcal{U}_* \mapsto \mathcal{U}_*$ ,  $\text{opt} \mapsto \text{crit}$ ,  $J$ -optimal  $\mapsto J$ -critical,  $\text{ARE} \mapsto [\text{e}] \text{CARE}$ ,  $\text{IRE} \mapsto \text{IARE}$ ,  $\mathcal{A}, \mathcal{B}, \dots \mapsto \mathbb{A}, \mathbb{B}, \dots$ ).

## 2 Well-posed linear systems (WPLSs)

If the generators of the system (1) are *bounded*, i.e.,  $[\frac{A}{C}|\frac{B}{D}] \in \mathcal{B}(H \times U, H \times Y)$ , then the unique solution of (1) is obviously given by the system

$$\begin{cases} x(t) &= \mathcal{A}^t x_0 + \mathcal{B}\tau^t u \\ y &= \mathcal{C}x_0 + \mathcal{D}u, \end{cases} \quad (10)$$

where

$$\begin{aligned} \mathcal{A}^t &= e^{At}, & \mathcal{B}\tau^t u &= \int_0^t \mathcal{A}^{t-s} B u(s) ds, \\ (\mathcal{C}x_0)(t) &= C \mathcal{A}^t x_0, & (\mathcal{D}u)(t) &= C \mathcal{B}\tau^t u + D u(t). \end{aligned} \quad (11)$$

This is illustrated in Figure 1.

The formulae (1), (10) and (11) are actually valid for rather unbounded generators. Therefore, the WPLSs are defined by requiring  $\mathcal{A}$  to be a strongly continuous semigroup,  $\mathcal{D}$  to be time-invariant and causal, and  $\mathcal{B}$  and  $\mathcal{C}$  to be compatible with  $\mathcal{A}$  and  $\mathcal{D}$ ; in addition, one requires that  $[\frac{\mathcal{A}^t}{\mathcal{C}}|\frac{\mathcal{B}\tau^t}{\mathcal{D}}]$  is linear and continuous  $H \times L_{\text{loc}}^2(\mathbb{R}_+; U) \rightarrow$

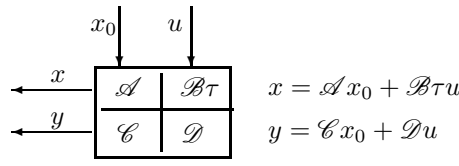


Figure 1: Input/state/output diagram of a WPLS  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$

$H \times L_{\text{loc}}^2(\mathbb{R}_+; Y)$  for each  $t \geq 0$ , equivalently, that

$$\|x(t)\|_H^2 + \int_0^t \|y(s)\|_Y^2 ds \leq K_t (\|x_0\|_H^2 + \int_0^t \|u(s)\|_U^2 ds) \quad (12)$$

for some (equivalently, all)  $t > 0$ , where  $K_t$  depends on  $t$  only. An equivalent formulation (due to Olof Staffans) is given in Definition 2.1, where we use the unique natural extensions of  $\mathcal{B}$  and  $\mathcal{D}$  that allow the inputs to be defined on the whole real line, thus simplifying several formulae.

We use the notation  $L_\omega^2 = e^{\omega \cdot} L^2 = \{f \mid e^{-\omega \cdot} f \in L^2\}$  (similarly,  $L_\omega^1 := e^{\omega \cdot} L^1$ ),  $(\tau^t u)(s) := u(t+s)$  and  $\pi_\pm u := \chi_{\mathbb{R}_\pm} u$ , where  $\chi_E(t) := \begin{cases} 1, & t \in E; \\ 0, & t \notin E. \end{cases}$  (Similarly,  $\pi_E u := \chi_E u$  when  $E \subset \mathbb{R}$ .) We also consider  $\pi_+$  as the projection  $L^2(\mathbb{R}; U) \rightarrow L^2(\mathbb{R}_+; U)$  or as its adjoint.

Throughout this article, we assume that  $\Sigma = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  is a WPLS on  $(U, H, Y)$ , i.e., that 1.–4. below hold for some  $\omega \in \mathbb{R}$ :

**Definition 2.1 (WPLS and stability)** *Let  $\omega \in \mathbb{R}$ . An  $\omega$ -stable well-posed linear system on  $(U, H, Y)$  is a quadruple  $\Sigma = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ , where  $\mathcal{A}^t$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  are bounded linear operators of the following type:*

1.  $\mathcal{A}^t : H \rightarrow H$  is a strongly continuous semigroup of bounded linear operators on  $H$  satisfying  $\sup_{t \geq 0} \|e^{-\omega t} \mathcal{A}^t\|_H < \infty$ ;
2.  $\mathcal{B} : L_\omega^2(\mathbb{R}; U) \rightarrow H$  satisfies  $\mathcal{A}^t \mathcal{B} u = \mathcal{B} \tau^t \pi_- u$  for all  $u \in L_\omega^2(\mathbb{R}; U)$  and  $t \in \mathbb{R}_+$ ;
3.  $\mathcal{C} : H \rightarrow L_\omega^2(\mathbb{R}; Y)$  satisfies  $\mathcal{C} \mathcal{A}^t x = \pi_+ \tau^t \mathcal{C} x$  for all  $x \in H$  and  $t \in \mathbb{R}_+$ ;
4.  $\mathcal{D} : L_\omega^2(\mathbb{R}; U) \rightarrow L_\omega^2(\mathbb{R}; Y)$  satisfies  $\tau^t \mathcal{D} u = \mathcal{D} \tau^t u$ ,  $\pi_- \mathcal{D} \pi_+ u = 0$ , and  $\pi_+ \mathcal{D} \pi_- u = \mathcal{C} \mathcal{B} u$  for all  $u \in L_\omega^2(\mathbb{R}; U)$  and  $t \in \mathbb{R}$ .

The different components of  $\Sigma = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  are named as follows:  $U$  is the input space,  $H$  the state space,  $Y$  the output space,  $\mathcal{A}$  the semigroup,  $\mathcal{B}$  the reachability map,  $\mathcal{C}$  the observability map, and  $\mathcal{D}$  the I/O map (input/output map) of  $\Sigma$ .

We say that  $\mathcal{A}$  (resp.  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$ ) is  $\alpha$ -stable if 1. (resp. 2., 3., 4.) holds for  $\omega = \alpha$ . Stable means 0-stable; exponentially stable means  $\omega$ -stable for some  $\omega < 0$ . The system is output stable (resp. SOS-stable) if  $\mathcal{C}$  (resp.  $\mathcal{C}$  and  $\mathcal{D}$ ) is stable. We set  $\Sigma^\tau := \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  (cf. (13)).

(A SOS (Stable-Output System) satisfies  $y \in L^2$  for all  $x_0 \in H$ ,  $u \in L^2$ , where  $y := \mathcal{C} x_0 + \mathcal{D} u$ .)

Any sub- or superscripts of a system are inherited by its parts and generators (see Lemma 2.4 and Definition 2.6); e.g.,  $\mathcal{A}_L, \mathcal{B}_L, \mathcal{C}_L, \mathcal{D}_L$  denote the maps and  $A_L, B_L, C_L, D_L$  the generators of  $\Sigma_L$  (in Lemma 3.1). Practically all conventions above and below follow [S04], [M02] etc.

Exponential stability of a system is equivalent to that of its semigroup, hence Datko's Theorem leads to the following:

**Lemma 2.2** *A WPLS is  $\omega$ -stable for any  $\omega > \omega_A := \inf_{t>0} [t^{-1} \log \|\mathcal{A}^t\|]$ . In particular, it is exponentially stable iff  $\mathcal{A} x_0 \in L^2(\mathbb{R}_+; H)$  for all  $x_0 \in H$ .  $\square$*

(See Lemmas 6.1.10(a1) and A.4.5 of [M02].)

**Definition 2.3 (State and output)** *With initial time zero, initial value  $x_0 \in H$ , and control (or input)  $u \in L^2_\omega(\mathbb{R}_+; U)$ , the controlled state  $x(t) \in H$  at time  $t \in \mathbb{R}_+$  and the output  $y \in L^2_\omega(\mathbb{R}_+, Y)$  of  $\Sigma$  are given by (cf. Figure 1)*

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{A}^t & \mathcal{B}\tau^t \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} \mathcal{A}^t x_0 + \mathcal{B}\tau^t u \\ \mathcal{C}x_0 + \mathcal{D}u \end{bmatrix}. \quad (13)$$

Sometimes we use the equivalent notation

$$\begin{bmatrix} \mathcal{A}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{D}^t \end{bmatrix} := \begin{bmatrix} \mathcal{A}^t & \mathcal{B}\tau^t \pi_{[0,t]} \\ \pi_{[0,t]} \mathcal{C} & \pi_{[0,t]} \mathcal{D} \pi_{[0,t]} \end{bmatrix} : \begin{bmatrix} x_0 \\ u \end{bmatrix} \mapsto \begin{bmatrix} x(t) \\ \pi_{[0,t]} y \end{bmatrix}. \quad (14)$$

G. Weiss et al. use symbols  $\begin{bmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{bmatrix} := \begin{bmatrix} \mathcal{A}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{D}^t \end{bmatrix}$  and a different but equivalent definition of WPLSs.

By causality, the state and output (in particular,  $\mathcal{D}$  and  $\mathcal{B}\tau$ ) are well defined for any  $u \in L^2_{\text{loc}}(\mathbb{R}_+; U)$  (with  $y \in L^2_{\text{loc}}(\mathbb{R}_+; Y)$ ), or even  $u \in L^2_\omega(\mathbb{R}; U) + L^2_{\text{loc}}(\mathbb{R}_+; U)$ .

The existence of a feedthrough operator (“ $D$ ”) is equivalent to regularity (Definition 2.6), but a WPLS always has generators  $\left[\frac{A}{C} \middle| \frac{B}{D}\right]$  that satisfy the rest of (11):

**Lemma 2.4** ( $A, B, C$ ) *Let  $A$  be the generator of  $\mathcal{A}$  and let  $\alpha \in \rho(A)$ .<sup>3</sup>*

*We set  $H_1 := \text{Dom}(A)$  with  $\|x\|_{H_1} := \|(\alpha - A)x\|_H$  (this is equivalent to the graph norm), and define  $H_{-1}$  to be the completion of  $H$  under the norm  $\|(\alpha - A)^{-1} \cdot\|_H$  (thus  $H_1 \subset H \subset H_{-1}$ ).*

*The following hold:*

- (a)  $\mathcal{A}$  can be isometrically extended to  $H_{-1}$  and restricted to  $H_1$ . We identify the three semigroups (“ $\mathcal{A}$ ”) and their generators (“ $A$ ”); thus, the map  $\alpha - A$  is an isometric isomorphism of  $H_n$  onto  $H_{n-1}$  ( $n = 0, 1$ ).
- (b) There is a unique input operator  $B \in \mathcal{B}(U, H_{-1})$  s.t. ( $u \in L^2_{\text{loc}}(\mathbb{R}_+; U)$ ,  $t \geq 0$ )

$$\mathcal{B}\tau^t u = \int_0^t \mathcal{A}^{t-s} B u(s) ds \in H \quad (15)$$

*(the integration is carried out in  $H_{-1}$  but the integral belongs to  $H$ ). Moreover,  $x := \mathcal{A}x_0 + \mathcal{B}\tau u$  satisfies  $x' = Ax + Bu$  in  $H_{-1}$  a.e. on  $\mathbb{R}_+$  and  $x(t) - x_0 = \int_0^t (Ax + Bu) dm$  for all  $t \geq 0$ ,  $x_0 \in H$ ,  $u \in L^2_{\text{loc}}(\mathbb{R}_+; U)$ .*

- (c) There is a unique output operator  $C \in \mathcal{B}(H_1, Y)$  s.t.

$$(\mathcal{C}x_0)(t) = C\mathcal{A}^t x_0 \quad (\forall x_0 \in H_1, \quad t \geq 0). \quad (16)$$

*Moreover,  $(\mathcal{C}x_0)(t) = C_w \mathcal{A}^t x_0$  for a.e.  $t > 0$  and all  $x_0 \in H$  (see (18) for  $C_w$ ).*

*We say that  $\Sigma$  is generated by  $\left[\frac{A}{C} \middle| \frac{B}{D}\right]$ , and we call  $\left[\frac{A}{C} \middle| \frac{B}{D}\right]$  the generators of  $\Sigma$ ; they are independent of  $\alpha$  (and  $\omega$ ). Also the following hold:*

- (d)  $\left[\frac{A}{C} \middle| \frac{B}{D}\right]$  determine  $\left[\frac{\mathcal{A}}{\mathcal{C}} \middle| \frac{\mathcal{B}}{\mathcal{D}}\right]$  uniquely and  $\mathcal{D}$  modulo an additive constant from  $\mathcal{B}(U, Y)$ .

We consider  $H$  as the pivot space, so that  $H_{-1} = \text{Dom}(A^*)^*$ ,  $B^* \in \mathcal{B}(\text{Dom}(A^*), U)$ , and  $C^* \in \mathcal{B}(Y, \text{Dom}(A)^*)$  (see Definition 6.1.17 of [M02] for details).

Let  $\omega \in \mathbb{R}$ . We define  $\text{TIC}_\omega(U, Y)$  to be the (closed) subspace of operators  $\mathcal{D} \in \mathcal{B}(L^2_\omega(\mathbb{R}; U); L^2_\omega(\mathbb{R}; Y))$  that are *causal* (i.e.,  $\pi_- \mathcal{D} \pi_+ = 0$ ) and *time-invariant*, i.e.  $\tau^t \mathcal{D} = \mathcal{D} \tau^t$  for all  $t \in \mathbb{R}$ . The I/O maps of WPLSs are exactly all such operators ( $\text{TIC}_\infty(U, Y) := \cup_{\omega \in \mathbb{R}} \text{TIC}_\omega(U, Y)$ , often called “the well-posed I/O maps”). In fact, they can be identified with proper transfer functions (i.e., functions bounded and holomorphic on some right half-plane, which we denote by  $H^\infty_\omega(U, Y) := \cup_{\omega \in \mathbb{R}} H^\infty(\mathbb{C}_\omega^+; \mathcal{B}(U, Y))$ ):

**Theorem 2.5 (Transfer functions  $\hat{\mathcal{D}}$ )** *For each  $\mathcal{D} \in \text{TIC}_\omega(U, Y)$ , there is a unique function  $\hat{\mathcal{D}} \in H^\infty(\mathbb{C}_\omega^+; \mathcal{B}(U, Y))$ , called the transfer function (or symbol) of  $\mathcal{D}$ , s.t.  $\hat{\mathcal{D}}u = \hat{\mathcal{D}}\hat{u}$  on  $\mathbb{C}_\omega^+$  for all  $u \in L^2_\omega(\mathbb{R}_+; U)$ . The mapping  $\mathcal{D} \mapsto \hat{\mathcal{D}}$  is an isometric isomorphism of  $\text{TIC}_\omega(U, Y)$  onto  $H^\infty(\mathbb{C}_\omega^+; \mathcal{B}(U, Y))$ .  $\square$*

<sup>3</sup>The exact value of  $\alpha$  is insignificant, since resulting norms on  $H_1$  or  $H_{-1}$  are equivalent, by the resolvent equation.



Here  $\mathcal{B}(U, Y)$  denotes the space of bounded linear operators  $U \rightarrow Y$ ,  $H^\infty(\mathbb{C}_\omega^+; \mathcal{B}(U, Y))$  denotes the Banach space of bounded holomorphic functions  $\mathbb{C}_\omega^+ \rightarrow \mathcal{B}(U, Y)$ , and  $\hat{u}$  denotes the Laplace transform of  $u$ :

$$\hat{u}(s) := \int_{\mathbb{R}} e^{-st} u(t) dt \quad (s \in \mathbb{C}_\omega^+ := \{s \in \mathbb{C} \mid \operatorname{Re} s > \omega\}). \quad (17)$$

If  $f$  is holomorphic on  $\mathbb{C}_\omega^+$ , and  $\Omega \subset \mathbb{C}_\omega^+$  is open, then we identify  $f$  and  $f|_\Omega$ . In fact, we do this whenever  $f$  is holomorphic on  $\mathbb{C}_\omega^+ \setminus E$ , where  $E$  does not have limit points on  $\mathbb{C}_\omega^+$ . Since any holomorphic extensions to right half-planes are unique, this does not cause problems (not even with  $E$  if we remove removable singularities).

A *realization* of  $\mathcal{D}$  or  $\hat{\mathcal{D}}$  means a WPLS whose I/O map is  $\mathcal{D}$ .

If  $\hat{\mathcal{D}}$  has a limit at infinity (along the positive real axis), then the system is called regular:

**Definition 2.6 (D, Regularity)** We call  $\mathcal{D} \in \operatorname{TIC}_\omega(U, Y)$  (and  $\hat{\mathcal{D}}$  and  $[\frac{\mathcal{A}}{\mathcal{C}} | \frac{\mathcal{B}}{\mathcal{D}}]$ ) weakly (resp. strongly, uniformly) regular (WR (resp. SR, UR)) with feedthrough operator  $\hat{\mathcal{D}}(+\infty) := D \in \mathcal{B}(U, Y)$  if  $\hat{\mathcal{D}}(s) \rightarrow D$  weakly (resp. strongly) as  $s \rightarrow +\infty$  on  $(\omega, +\infty)$ .

We call  $\mathcal{D}$  ULR (uniformly line-regular) if  $\|\hat{\mathcal{D}}(s) - D\| \rightarrow 0$  as  $\operatorname{Re} s \rightarrow +\infty$  (uniformly with respect to  $\operatorname{Im} s$ ).

If  $\Sigma$  is WR, then we say that  $[\frac{A}{C} | \frac{B}{D}]$  are the *generators* of  $\Sigma$ , since they determine the system uniquely, and we sometimes denote  $\Sigma$  by  $(\frac{A}{C} | \frac{B}{D})$ . Any WPLS with *bounded*  $B$  or  $C$  (i.e.,  $B \in \mathcal{B}(U, H)$  or  $C \in \mathcal{B}(H, Y)$ ) is ULR, by Lemma 6.3.16 of [M02]. An equivalent condition for the weak regularity of  $\Sigma$  is that  $(\alpha - A)^{-1}BU \subset \operatorname{Dom}(C_w)$ , where

$$\operatorname{Dom}(C_w) := \{x \in H \mid C_w x := \lim_{s \rightarrow +\infty} C s(s - A)^{-1}x \text{ exists}\}. \quad (18)$$

(Here w-lim is the weak limit (in  $Y$ ). The above condition is independent of  $\alpha \in \rho(A)$ .) The map  $C_w : \operatorname{Dom}(C_w) \rightarrow Y$  is called the *weak Yosida extension* of  $C$ . If  $\Sigma$  is WR and  $\omega$ -stable, then  $\hat{\mathcal{D}}(s) = D + C_w(s - A)^{-1}B$  when  $\operatorname{Re} s > \omega$ , and  $y = C_w x + Du$  a.e. for all  $x_0 \in H$  and all  $u \in L^2_{\operatorname{loc}}(\mathbb{R}_+; U)$ . Similar claims hold for  $C_s$ , s-lim and “SR”.

Using Lemma 2.4, one can show that any  $[\frac{A}{C} | \frac{B}{D}] \in \mathcal{B}(H \times U, H_{-1} \times Y)$  are the generators of a WR WPLS iff  $[\frac{\mathcal{A}^t}{\mathcal{C}} | \frac{\mathcal{B}^t}{\mathcal{D}}]$  defined by (11) a.e. (with  $C_w$  in place of  $C$ ) are bounded  $H \times L^2([0, t]; U) \rightarrow H \times L^2([0, t]; Y)$  for some (hence all)  $t > 0$ . In (11), “ $\mathcal{A}^t = e^{At}$ ” must be interpreted as the requirement that  $A$  generates a  $C_0$ -semigroup  $\mathcal{A}$ .

The dual system  $(\frac{A^*}{B^*} | \frac{C^*}{D^*})$  can be defined for arbitrary WPLSs:

**Lemma 2.7 (Dual system  $\Sigma^d$ )** If  $\Sigma$  is an  $\omega$ -stable WPLS, then so is its dual system

$$\Sigma^d := \left[ \begin{array}{c|c} \mathcal{A}^d & \mathcal{C}^d \\ \hline \mathcal{B}^d & \mathcal{D}^d \end{array} \right] := \left[ \begin{array}{c|c} \mathcal{A}^* & \mathcal{C}^* \mathcal{R} \\ \hline \mathcal{R} \mathcal{B}^* & \mathcal{R} \mathcal{D}^* \mathcal{R} \end{array} \right] \quad (19)$$

(over  $(Y, H, U)$ ), where  $(\mathcal{R}u)(t) := u(-t)$ . Moreover,  $(\Sigma^d)^d = \Sigma$ , and  $[\frac{A^*}{B^*} | \frac{C^*}{D^*}]$  ( $[\frac{A^*}{B^*} | \frac{C^*}{D^*}]$  if  $\Sigma$  is WR) are the generators of  $\Sigma^d$ , and  $\hat{\mathcal{D}}^d(s) = \hat{\mathcal{D}}(\bar{s})^* \forall s \in \mathbb{C}_\omega^+$ .  $\square$

(This is well-known, see Lemmas 6.1.4, 6.2.2 and 6.2.9(b) of [M02].) We use  $L^2$  as the pivot space (p. 898 of [M02]); e.g.,  $\int_{\mathbb{R}} \langle \mathcal{C} x_0, \tilde{y} \rangle(t) dt = \langle x_0, \mathcal{C}^* \tilde{y} \rangle_H$ . Thus,  $\Sigma^d$  is independent of  $\omega$  (and  $\mathcal{C} \in \mathcal{B}(H, L^2_\omega(\mathbb{R}; Y)) \Leftrightarrow \mathcal{C}^* \in \mathcal{B}(L^2_{-\omega}(\mathbb{R}; Y), H)$ ).

**Notes for Section 2:** Everything in this section is well known; see, e.g., [W94a] and [W94b] (or Sections 6.1–6.2 of [M02]). Much more on WPLSs can be found in [M02] too, but [S04] is the most thorough book on the subject and also covers  $L^p$  signals for  $p \neq 2$  and for general Banach spaces in place of  $U, H, Y$ .

The Lax–Phillips scattering theory and the operator-based model theory of Béla Sz. Nagy and Ciprian Foiaş have been shown equivalent to WPLSs (see Chapter 11 of [S04]). The former has been extensively developed in the (ex-) Soviet Union area by Damir Z. Arov and others (cf. [AN96]), independently of WPLSs. See pp. 23 and 167 of [M02] for further details and references.

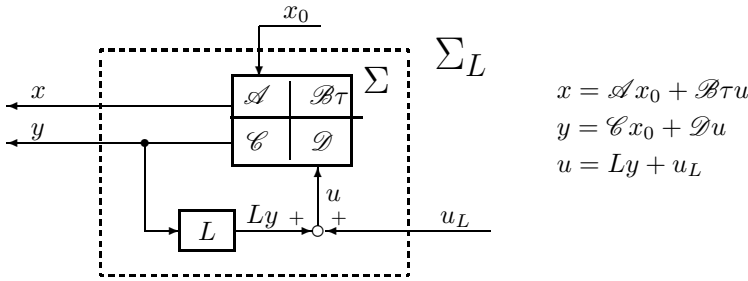


Figure 2: Static output feedback

### 3 State feedback

In this section we first define (static) output feedback (Lemma 3.1). Then we extend state feedback (the formula  $u(t) = Kx(t)$ ) to WPLSs, first in a “generalized” sense (Definition 3.2) and then in the standard sense (Definition 3.5). For the former one can more easily generalize Theorems 1.1 and 1.3, but the latter is more desirable in the applications.

Output feedback means feeding the output  $y$  back to the input  $u$  through some feedback operator  $L \in \mathcal{B}(Y, U)$ , i.e.,  $u = Ly + u_L$ , where  $u_L$  is the external input, as in Figure 2. Obviously, the closed loop formulas  $\begin{bmatrix} \mathcal{A}_L & \mathcal{B}_L\tau \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix} : \begin{bmatrix} x_0 \\ u_L \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}$  can be uniquely solved iff  $I - L\mathcal{D}$  is invertible (equivalently,  $I - L\mathcal{D} \in \mathcal{GTIC}_\infty$ ). The solution is the following:

**Lemma 3.1** ( $\Sigma_L$ ) *Let  $L \in \mathcal{B}(Y, U)$  be an admissible output feedback operator for  $\Sigma$  (i.e.,  $I - L\mathcal{D} \in \mathcal{GTIC}_\infty(U)$ ). Then also the closed-loop system  $\Sigma_L$  is a WPLS over  $(U, H, Y)$ , where*

$$\Sigma_L := \begin{bmatrix} \mathcal{A}_L & \mathcal{B}_L \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix} := \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau L(I - \mathcal{D}L)^{-1}\mathcal{C} & \mathcal{B}(I - L\mathcal{D})^{-1} \\ (I - \mathcal{D}L)^{-1}\mathcal{C} & \mathcal{D}(I - L\mathcal{D})^{-1} \end{bmatrix}. \quad (20)$$

□

(See, e.g., Section 6 of [W94b] for the proof.)

Next we define an important generalized form of state feedback. Given a WPLS and a control law  $\mathcal{K}_0 : x_0 \mapsto u$ , the corresponding function  $x_0 \mapsto \begin{bmatrix} x \\ y \\ u \end{bmatrix}$  is called a controlled WPLS form iff it is (the left column of) a WPLS (equivalently, iff  $\begin{bmatrix} \mathcal{A}_0 \\ \mathcal{C}_0 \\ \mathcal{K}_0 \end{bmatrix} : x_0 \mapsto \begin{bmatrix} x \\ y \\ u \end{bmatrix}$  is):

**Definition 3.2** ( $\mathcal{K}_0, \Sigma_0$ , **WPLS form**) *We call the control  $x_0 \mapsto \mathcal{K}_0 x_0$  a control for  $\Sigma$  in WPLS form (and  $\Sigma_0$  a controlled WPLS form for  $\Sigma$ ) if  $\mathcal{K}_0 : H \rightarrow L^2_{\text{loc}}(\mathbb{R}_+; U)$  is s.t.  $\Sigma_0$  is a WPLS<sup>4</sup> (on  $(\{0\}, H, Y \times U)$ ), where*

$$\Sigma_0 := \begin{bmatrix} \mathcal{A}_0 & | \\ \mathcal{C}_0 & | \\ \mathcal{K}_0 & | \end{bmatrix} := \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau\mathcal{K}_0 & | \\ \mathcal{C} + \mathcal{D}\mathcal{K}_0 & | \\ \mathcal{K}_0 & | \end{bmatrix}. \quad (21)$$

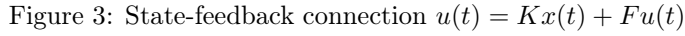
A control in WPLS form need not be of (well-posed) state-feedback form unless, e.g.,  $B$  is bounded (see p. 374 of [M02]). However, it can be considered as being of non-well-posed state-feedback form, since  $u(t) = (K_0)_w x(t)$  a.e., by, e.g., (5.6) of [W94b].

Controls in WPLS form can be easily characterized in the frequency domain too:

**Lemma 3.3** ( $\Sigma_0$ ) *A triple  $\Sigma_0 := \begin{bmatrix} \mathcal{A}_0 \\ \mathcal{C}_0 \\ \mathcal{K}_0 \end{bmatrix}$  is a controlled WPLS form for  $\Sigma$  iff there exist  $\omega \in \mathbb{R}$  and linear operators  $A_0$  on  $H$  and  $K_0 : \text{Dom}(A_0) \rightarrow U$  s.t.  $\mathcal{K}_0 \in \mathcal{B}(H, L^2_\omega(\mathbb{R}_+; U))$ ,  $\mathcal{C}_0 = \mathcal{C} + \mathcal{D}\mathcal{K}_0$ ,  $\mathcal{A}_0 = \mathcal{A} + \mathcal{B}\tau\mathcal{K}_0$ ,  $\begin{bmatrix} \mathcal{A}_0 x_0 \\ \mathcal{C}_0 x_0 \end{bmatrix} (s) = \begin{bmatrix} I \\ K_0 \end{bmatrix} (s - A_0)^{-1} x_0 \ \forall x_0 \in H \ \forall s \in \mathbb{C}_\omega^+$ .*

**Proof:** “Only if” is quite obvious, so we prove “if”. Assume, w.l.o.g., that  $\Sigma$  is  $\omega$ -stable (increase  $\omega$  if necessary). One easily verifies that  $\mathcal{C}_0 \in \mathcal{B}(H, L^2_\omega(\mathbb{R}_+; Y))$ ,  $\mathcal{A}_0 x_0 \in \mathcal{C}(\mathbb{R}_+; H)$ ,  $\mathcal{A}_0^t \in \mathcal{B}(H)$  ( $t \geq 0$ ),  $\mathcal{A}_0^0 = I$ ,  $\|\mathcal{A}_0^t\| \leq Me^{\omega t}$  (use (2.2) of [M02]). By Lemma B.5,  $\mathcal{A}_0$  is a semigroup. By Lemma 6.3.15 of [M02],  $\begin{bmatrix} \mathcal{A}_0 \\ \mathcal{K}_0 \end{bmatrix}$  is a WPLS (note that

<sup>4</sup>Like here, we sometimes omit a zero input column (or output row) from a WPLS.


$$= \mathcal{C} \mathcal{A}^t + \pi_+ \mathcal{D} \mathcal{K}_0 \mathcal{A}_0^t + \mathcal{C} \mathcal{B} \tau^t \mathcal{K}_0 = \mathcal{C} \mathcal{A}_0^t + \mathcal{D} \mathcal{K}_0 \mathcal{A}_0^t = \mathcal{C}_0 \mathcal{A}_0^t. \quad (23)$$

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loop system (see Figure 3)

$$\Sigma_{\circ}^{\tau} = \left[ \begin{array}{c|c} \mathcal{A}_{\circ} & \mathcal{B}_{\circ}\tau \\ \hline \mathcal{C}_{\circ} & \mathcal{D}_{\circ} \\ \hline \mathcal{K}_{\circ} & \mathcal{F}_{\circ} \end{array} \right] = \left[ \begin{array}{c|c} \mathcal{A} + \mathcal{B}\tau\mathcal{M}\mathcal{K} & \mathcal{B}\mathcal{M}\tau \\ \hline \mathcal{C} + \mathcal{D}\mathcal{M}\mathcal{K} & \mathcal{D}\mathcal{M} \\ \hline \mathcal{M}\mathcal{K} & \mathcal{M} - I \end{array} \right] \quad (25)$$

$$= \Sigma_{\text{ext}}^{\tau} \begin{bmatrix} I & 0 \\ -\mathcal{K} & I - \mathcal{F} \end{bmatrix}^{-1} = \Sigma_{\text{ext}}^{\tau} \begin{bmatrix} I & 0 \\ \mathcal{M}\mathcal{K} & \mathcal{M} \end{bmatrix} : \begin{bmatrix} x_0 \\ u_{\circ} \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ u - u_{\circ} \end{bmatrix}. \quad (26)$$

If  $\mathcal{F}$  is weakly regular and  $F = 0$ , then we call the generator  $K$  (or  $K_w$ ) of  $\mathcal{K}$  a weakly regular state-feedback operator for  $\Sigma$ .

We call  $[\mathcal{K} | \mathcal{F}]$  stabilizing if  $\Sigma_{\circ}$  is stable. We add “[q./r.c.-” if  $\mathcal{N}, \mathcal{M}$  are [q./r.c. (Definition 5.4 below). If there exists a stabilizing state-feedback pair for  $\Sigma$ , then  $\Sigma$  is called stabilizable (similarly for exponentially, SOS- or output-stabilizing etc.).

Obviously,  $I - \mathcal{F} \in \mathcal{GTIC}_{\infty}(U)$  iff  $L = [I \ 0]$  is an admissible output feedback operator for  $\Sigma_{\text{ext}}$ . By Lemma 3.1,  $\Sigma_{\circ}$  is then indeed a WPLS (on  $(U, H, Y \times U)$ ). If  $\mathcal{D}$  and  $\mathcal{F}$  are strongly regular with feedthrough operators  $D$  and  $F = 0$ , then the generators of the two systems are as follows:

$$\Sigma_{\text{ext}} = \left( \begin{array}{c|c} A & B \\ \hline C & D \\ \hline K & 0 \end{array} \right), \quad \Sigma_{\circ} = \left( \begin{array}{c|c} A + BK & B \\ \hline C + DK & D \\ \hline K & 0 \end{array} \right), \quad (27)$$

by Proposition 6.6.18(d4) of [M02] (or [W94b]). Observe that  $\Sigma_{\circ} \begin{bmatrix} I \\ 0 \end{bmatrix}$  is a controlled WPLS form.

We can reduce most output feedback results to dynamic feedback results:

**Remark 3.6** (“ $\Sigma_{\circ} = \Sigma_L$ ”) Any static output feedback can be written as (part of) state feedback and vice versa.

**Proof:** We observed above that the state feedback  $[\mathcal{K} | \mathcal{F}]$  for  $\Sigma$  corresponds to the static output feedback  $L = [0 \ I]$  for  $\Sigma_{\text{ext}}$ , i.e.,  $\Sigma_{\circ} = (\Sigma_{\text{ext}})_I$ . Conversely, static output feedback can be written as a special case of state feedback (set  $[\mathcal{K} | \mathcal{F}] = [L\mathcal{C} \mid L\mathcal{D}]$  and drop the bottom row of  $\Sigma_{\circ}$  to obtain  $\Sigma_L$ ).

Moreover, given a WPLS  $\Sigma = [\frac{\mathcal{A}}{\mathcal{C}} | \frac{\mathcal{B}}{\mathcal{D}}]$ , a pair  $[\mathcal{K} | \mathcal{F}]$  is admissible for  $\Sigma$  iff  $[\mathcal{K} | \mathcal{F}]$  is admissible for  $[\mathcal{A} | \mathcal{B}]$ , and this is the case iff  $[\mathcal{K} | 0 \ \mathcal{D}]$  is an admissible state-feedback pair for  $[\mathcal{A} \mid 0 \ \mathcal{B}]$  (the closed-loop system equals  $\Sigma_{\circ}$  with a column of zeros inserted to the middle).

(Similarly, a “flow inverse” (see, e.g., Section 6.2 of [S04]) of  $[\frac{\mathcal{A}}{-\mathcal{K}} | \frac{\mathcal{B}}{\mathcal{F}}]$  means  $[\frac{\mathcal{A}_{\circ}}{\mathcal{K}_{\circ}} | \frac{\mathcal{B}_{\circ}}{\mathcal{F}_{\circ} + I}] = [\frac{\mathcal{A} + \mathcal{B}\mathcal{K}_{\circ}}{\mathcal{M}\mathcal{K}} | \frac{\mathcal{B}\mathcal{M}}{\mathcal{M}}]$ .)  $\square$

The state-feedback map  $\mathcal{K}_{\circ}$  determines the pair  $[\mathcal{K} | \mathcal{F}]$  uniquely modulo  $E \in \mathcal{GB}(U)$ :

**Lemma 3.7** (All  $[\mathcal{K} | \mathcal{F}]$ ) Let  $[\mathcal{K} | \mathcal{F}]$  be an admissible state-feedback pair for  $\Sigma$ . Then all admissible state-feedback pairs  $[\tilde{\mathcal{K}} | \tilde{\mathcal{F}}]$  leading to same control  $\mathcal{K}_{\circ}$  are given by

$$[\tilde{\mathcal{K}} | \tilde{\mathcal{F}}] = [E\mathcal{K} \mid I - E(I - \mathcal{F})] \quad (E \in \mathcal{GB}(U)). \quad (28)$$

Mnemonic:  $\tilde{\mathcal{K}} = E\mathcal{K}$ ,  $\tilde{\mathcal{F}} = E\mathcal{F}$ , where  $\mathcal{X} := \mathcal{M}^{-1}$ .

The following follows from a straight-forward computation:

**Lemma 3.8** Let  $[\mathcal{K} | \mathcal{F}]$  be an admissible state-feedback pair for  $\Sigma$ . Then  $x_{\circ} = x$  and  $y_{\circ} = y$  for any  $x_0 \in H$  and  $u \in L^2_{\text{loc}}(\mathbb{R}_+; U)$  if  $u_{\circ} = -\mathcal{K}x_0 + \mathcal{X}u$ , equivalently, if  $u = \mathcal{K}_{\circ}x_0 + \mathcal{M}u_{\circ}$ .

Moreover,  $\mathcal{K}_{\circ}$  is a control in WPLS form for  $[\frac{\mathcal{A}_{\circ}}{\mathcal{C}_{\circ}} | \frac{\mathcal{B}_{\circ}}{\mathcal{D}_{\circ}}]$  iff  $\mathcal{K}_0 := \mathcal{K}_{\circ} + \mathcal{M}\mathcal{K}_{\circ}$  is a control in WPLS form for  $\Sigma$ . We have  $\mathcal{K}_{\circ} = -\mathcal{K} + \mathcal{X}\mathcal{K}_0$ .  $\square$

Here  $x_\circ := \mathcal{A}_\circ x_0 + \mathcal{B}\tau u_\circ$  and  $y_\circ := \mathcal{C}_\circ x_0 + \mathcal{D}_\circ u_\circ$  are the state and output of  $\Sigma_\circ$  with input  $u_\circ \in L^2_{\text{loc}}(\mathbb{R}_+; U)$  and initial state  $x_0$ .

**Notes for Section 3:** Definition 3.2 and Lemma 3.3 are from [M02], and they are necessary tools for a complete Riccati equation theory, as shown by Theorem 7.1 (and Example 8.4.13 of [M02], cf. “3c.” on p. 67). Similar structures have implicitly been used in, e.g., [FLT88] and [Z96]. The rest of this section is essentially well known, mostly due to [W94b] (and [S98a]); see [S04] or [M02] for further results.

## 4 Optimal control and $J$ -coercivity

We shall present our main results on stabilization and factorization in Section 5, the ARE theory in Section 6, and the IRE theory in Section 7, thus generalizing the results of Section 1.

In this section (from Chapter 8 of [M02]) we present the optimization setting and certain tools on which those results are based. First we need to generalize “minimal” or “optimal” (“ $J$ -optimal”) control so as to cover 1. all WPLSs, 2. alternative optimization domains to  $\mathcal{U}_{\text{exp}}$  (29), and also 3. indefinite problems ( $\mathcal{J}(0, \cdot) \not\geq 0$ ). Then we need a general coercivity assumption on cost functions (“ $J$ -coercivity”) that covers much more (2) and (8) (in fact, all nonsingular control problems) and yet guarantees the existence of a unique optimal control (under the corresponding FCC).

A reader who wants to avoid technical details may consider  $B$ ,  $C$  and  $D$  bounded (so that they constitute a WPLS with any  $C_0$ -semigroup  $\mathcal{A}$  on  $H$ ) and  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  (and  $J = I$ ), as in Section 1. Hypothesis 4.1 is redundant in that setting.

In Theorem 1.1 we optimized over the set

$$\mathcal{U}_{\text{exp}}(x_0) := \{u \in L^2(\mathbb{R}_+; U) \mid x \in L^2(\mathbb{R}_+; H)\}, \quad (29)$$

sometimes called the set of exponentially (or internally) stabilizing controls (see (13) for  $x = x_{x_0, u}$ ). Recently it has become popular to study optimization over a larger set of controls than  $\mathcal{U}_{\text{exp}}(x_0)$ , namely over the set

$$\mathcal{U}_{\text{out}}(x_0) := \{u \in L^2(\mathbb{R}_+; U) \mid y \in L^2(\mathbb{R}_+; H)\} \quad (30)$$

of (externally or) output-stabilizing controls, as in Theorem 1.3. By discretization (Lemma 7.2 of [WR00]), one can show that  $\mathcal{U}_{\text{exp}} \subset \mathcal{U}_{\text{out}}$  (this means that  $\mathcal{U}_{\text{exp}}(x_0) \subset \mathcal{U}_{\text{out}}(x_0)$  for all  $x_0 \in H$ ), i.e., that  $u, x \in L^2 \Rightarrow y \in L^2$ . Sometimes the set  $\mathcal{U}_{\text{str}}(x_0) := \{u \in \mathcal{U}_{\text{out}}(x_0) \mid \|x(t)\|_H \rightarrow 0 \text{ as } t \rightarrow +\infty\}$  of strongly stabilizing controls is used, and in certain proofs (see, e.g., Corollaries 5.3 and 5.16, [M02], [M03b]) we need a very special domain of optimization. In general, we shall denote the chosen domain of optimization by  $\mathcal{U}_*(\cdot)$  (which a reader who wants to avoid technical details may read as  $\mathcal{U}_{\text{exp}}(\cdot)$ ).

**Standing Hypothesis 4.1 ( $\Sigma, J, \mathcal{U}_*, x, y$ )** *Throughout this article, we assume that  $\Sigma = [\frac{\mathcal{A}}{\mathcal{C}} + \frac{\mathcal{B}}{\mathcal{D}}]$  is a WPLS on Hilbert spaces  $(U, H, Y)$ , that  $J = J^* \in \mathcal{B}(Y)$ , and that  $\mathcal{U}_*$  and its parameters  $\vartheta, \mathcal{Q}, \mathcal{R}, Z^s, Z^u$  are of the form explained in the following paragraph. For any  $u, x_0$ , we define  $x, y$  by (13).*

We assume that  $\vartheta \in \mathbb{R}$ ,  $[\frac{\mathcal{A}}{\mathcal{D}} + \frac{\mathcal{B}}{\mathcal{R}}]$  is a WPLS on  $(U, H, \tilde{Y})$  for some Hilbert space  $\tilde{Y}$ ;  $Z^s$  is a Banach space and  $Z^u$  is a topological vector space (e.g., a normed space);  $Z^s \subset Z^u$  continuously;  $\mathcal{Q} \in \mathcal{B}(H, Z^u)$ ,  $\mathcal{R} \in \mathcal{B}(L^2(\mathbb{R}_+; U), Z^u)$ ; and  $\pi_+ \tau^t z \in Z^s \Leftrightarrow z \in Z^s$  ( $z \in Z^u$ ,  $t > 0$ ). Moreover, we set

$$\mathcal{U}_*(x_0) := \mathcal{U}_*^\Sigma(x_0) := \{u \in L^2_\vartheta(\mathbb{R}_+; U) \mid [\frac{\mathcal{C}}{\mathcal{D}} \frac{\mathcal{Q}}{\mathcal{R}}] \begin{bmatrix} x_0 \\ u \end{bmatrix} \in L^2 \times Z^s\}. \quad (31)$$

Thus, Standing Hypothesis 4.1 equals Hypothesis 9.0.1 of [M02] (plus (13)). Note that we can make  $\mathcal{U}_*$  equal to  $\mathcal{U}_{\text{exp}}$  (resp. to  $\mathcal{U}_{\text{out}}$ ) by setting  $\mathcal{Q} = \mathcal{A}$ ,  $\mathcal{R} = \mathcal{B}\tau$  (resp.  $\mathcal{Q} = \mathcal{C}$ ,  $\mathcal{R} = \mathcal{D}$ ),  $\tilde{Y} = H$ ,  $Z^s = L^2$ ,  $\vartheta = 0$ . The following is obvious:

**Lemma 4.2**  $\mathcal{U}_*(\alpha x_0 + \beta x_1) = \alpha \mathcal{U}_*(x_0) + \beta \mathcal{U}_*(x_1)$  whenever  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ ,  $\mathcal{U}_*(x_0) \neq \emptyset$ .

□

See Section 8.3 (and 9.0) of [M02] for further details and results.

Formula (13) is equivalent to (10) as well as to  $\begin{bmatrix} x(t) \\ \pi_{[0,t)} y \end{bmatrix} = \begin{bmatrix} \mathcal{A}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{D}^t \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix}$  (see (14)).

For any optimization domain  $\mathcal{U}_*$ , the results are quite similar to those for  $\mathcal{U}_{\text{exp}}$ . The main difference is that, instead of the exponential stability, we must require some other kind of stability for the closed-loop semigroup if  $\mathcal{U}_* \neq \mathcal{U}_{\text{exp}}$  (cf. Theorem 1.1(ii)). Further details will be given in Sections 6 and 7.

We want to have our theory applicable to any quadratic cost functions, hence we define the *cost function* by

$$\mathcal{J}(x_0, u) := \langle y, Jy \rangle_{L^2(\mathbb{R}_+; Y)} \quad (x_0 \in H, u \in \mathcal{U}_{\text{exp}}(x_0)). \quad (32)$$

If, e.g., we want to study the cost function (8) or  $\mathcal{J}(x_0, u) = \|y\|_2^2 + \|u\|_2^2$ , given a system  $\Sigma$ , we can achieve this by taking  $J = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \in \mathcal{B}(Y \times U)$  and replacing  $\mathcal{C}$  by  $\begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix}$  and  $\mathcal{D}$  by  $\begin{bmatrix} \mathcal{D} \\ I \end{bmatrix}$  (and  $Y$  by  $Y \times U$ ), as in the proof of Theorem 5.9. See the proof of Corollary 5.2 for the cost function (2).

Optimization theory is needed, e.g., in minimization problems (as in Theorems 1.1 and 1.3) and in  $H^\infty$  control problems, where  $J$  is indefinite and a saddle point of the cost function is sought (“best control for the worst disturbance”), since it leads to a formula for the desired controller. Since a minimum or a saddle point is necessarily a (often unique) zero of the derivative of the cost function, in such problems the goal is to find a control that is optimal in the following sense:

**Definition 4.3 ( $J$ -optimal)** *Let  $x_0 \in H$ . A control  $u \in \mathcal{U}_*(x_0)$  is called  $J$ -optimal for  $x_0$  if the Fréchet derivative (on  $\mathcal{U}_*(x_0)$ ) of the cost function  $u \mapsto \langle y, Jy \rangle$  is zero at  $u$ .*

*[Generalized] state-feedback  $[\mathcal{K} \mid \mathcal{F}]$  or  $K$  [or  $\mathcal{K}_0$  or  $\Sigma_0$ ] (Definition 3.5 [3.2]) is called  $J$ -optimal if, for all  $x_0 \in H$ , the control  $\mathcal{K}_\zeta x_0$  [or  $\mathcal{K}_0 x_0$ ] is  $J$ -optimal for  $x_0$ .*

By Lemma 4.4(b),  $u$  is  $J$ -optimal iff  $\langle \mathcal{D}\eta, Jy \rangle_{L^2} = 0$  (i.e., “ $\langle \Delta y, Jy \rangle_{L^2} = 0$ ”) for all  $\eta \in \mathcal{U}_*(0)$  (recall that  $y := \mathcal{C}x_0 + \mathcal{D}u$ ).

The optimal cost is always unique, although an optimal control might be nonunique:

**Lemma 4.4 (Optimal cost  $\mathcal{J}(x_0, u_{\text{opt}})$ )** (a) *If  $u$  and  $\tilde{u}$  are  $J$ -optimal for  $x_0 \in H$ , then the cost  $\langle y, Jy \rangle$  is the same for both  $y = \mathcal{C}x_0 + \mathcal{D}u$  and for  $\tilde{y} = \mathcal{C}x_0 + \mathcal{D}\tilde{u}$ .*

(b) *For any  $x_0 \in H$  and  $u \in \mathcal{U}_*(x_0)$ , the following are equivalent:*

- (i)  *$u$  is  $J$ -optimal for  $x_0$ ;*
- (ii)  *$\langle \mathcal{D}\eta, Jy \rangle = 0 \quad \forall \eta \in \mathcal{U}_*(0)$ ;*
- (iii)  *$\mathcal{J}(x_0, u + \eta) = \langle y, Jy \rangle + \langle \mathcal{D}\eta, J\mathcal{D}\eta \rangle \quad \forall \eta \in \mathcal{U}_*(0)$ ;*
- (iv)  *$\langle \mathcal{C}\tilde{x}_0 + \mathcal{D}(\tilde{u} + \tilde{\eta}), J(\mathcal{C}x_0 + \mathcal{D}(u + \eta)) \rangle = \langle \tilde{y}, Jy \rangle + \langle \mathcal{D}\tilde{\eta}, J\mathcal{D}\eta \rangle$  whenever  $\eta, \tilde{\eta} \in \mathcal{U}_*(0)$  and  $\tilde{u}$  is  $J$ -optimal for  $\tilde{x}_0 \in H$ ,  $\tilde{y} := \mathcal{C}\tilde{x}_0 + \mathcal{D}\tilde{u}$ .*

(c) *If there is at most one  $J$ -optimal control for  $x_0 = 0$ , then there is at most one  $J$ -optimal control for any  $x_0 \in H$ .*

(d) *If (f)  $\mathcal{J}(0, \cdot) \geq 0$ , then a control is minimizing iff it is  $J$ -optimal.* □

**Proof:** (a) We have  $\langle \tilde{y}, J\tilde{y} \rangle - \langle y, Jy \rangle = \langle Jy, \tilde{y} - y \rangle + \langle \tilde{y} - y, J\tilde{y} \rangle = 0$ , because  $\tilde{y} - y = \mathcal{D}\eta$ , where  $\eta := \tilde{u} - u \in \mathcal{U}_*(0)$ .

(b) Obviously, “(ii’)  $\text{Re} \langle \mathcal{D}\eta, Jy \rangle = 0$  for all  $\eta \in \mathcal{U}_*(0)$ ” is equivalent to (ii) and to (iii) (use  $i\eta$ ). But  $\frac{d\mathcal{J}(x_0, u_{\text{opt}} + t\eta)}{dt}(0) = 2 \text{Re} \langle y, J\mathcal{D}\eta \rangle$ , hence (ii’) is equivalent to (i). Trivially, (iv) implies (iii); conversely, by using (iii) three times to compute  $\mathcal{J}(x_0 + \tilde{x}_0, u + \tilde{u} + \eta + \tilde{\eta})$ , we obtain  $2 \text{Re}(\text{iv})$ , hence then (iv) holds.

(c) If  $u, u + \eta$  are  $J$ -optimal for  $x_0$ , then  $\eta$  is  $J$ -optimal for 0, by (iv) (set  $\tilde{x}_0 = 0$ ,  $\tilde{u} = 0$ ).

(d) This follows from (iii), because  $\mathcal{J}(0, \eta) = \langle \mathcal{D}\eta, J\mathcal{D}\eta \rangle$ . □

To define  $J$ -coercivity, we need a natural norm on  $\mathcal{U}_*(0)$ :

**Lemma 4.5** ( $\|\cdot\|_{\mathcal{U}_*}$ ) *The set  $\mathcal{U}_*(0)$  is a Banach space under the norm  $\|u\|_{\mathcal{U}_*} := \max\{\|u\|_{L^2_\mathcal{D}}, \|\mathcal{D}u\|_2, \|\mathcal{R}u\|_{Z^s}\}$ .* □

(This is straight-forward, because  $L_\theta^2, L^2, Z^s$  are Banach spaces.)

Obviously, the norms  $\|u\|_{\mathcal{U}_{\text{exp}}}^2 := \|u\|_2^2 + \|x\|_2^2$  and  $\|u\|_{\mathcal{U}_{\text{out}}}^2 := \|u\|_2^2 + \|y\|_2^2$  are equivalent to those defined in (a) above. Moreover,  $\mathcal{D} \in \mathcal{B}(\mathcal{U}_*(0), L^2(\mathbb{R}_+; Y))$  (and  $\mathcal{B}\tau \in \mathcal{B}(\mathcal{U}_{\text{exp}}(0), L^2(\mathbb{R}_+; H))$ ).

In many of our results, we shall require that the *Popov Toeplitz operator*  $\mathcal{S}_{\text{PT}} := \mathcal{D}^* J \mathcal{D}$  is (boundedly) invertible  $\mathcal{U}_*(0) \rightarrow \mathcal{U}_*(0)^*$ . In the case  $J = I$ , this is true iff there exists  $\epsilon > 0$  s.t.  $\|\mathcal{D}u\|_2 \geq \epsilon \|u\|_{\mathcal{U}_*(0)}$  for all  $u \in \mathcal{U}_*(0)$ ; for  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  we can equivalently write this as  $\|\mathcal{D}u\|_2 \geq \epsilon' \|u\|_2 \forall u \in \mathcal{U}_{\text{out}}(0)$ .

This generalizes all coercivity assumptions that we have met in the literature (except those for “singular control”), including so called “no transmission zeros” and “no invariant zeros” conditions (see Theorem 11.2).

If  $\Sigma$  is exponentially stable and  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  (or  $\Sigma$  is SOS-stable and  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$ ), then an equivalent condition is that  $\mathcal{D}^* J \mathcal{D}$  is (boundedly) invertible on  $L^2(\mathbb{R}_+; U)$ , equivalently, that the *Popov function*  $\hat{\mathcal{D}}^* J \hat{\mathcal{D}}$  is uniformly invertible on the imaginary axis  $i\mathbb{R}$ .

See the proofs of the corollaries in Section 5 to observe how our condition is satisfied in various applications. E.g., for the “LQR” cost function  $\mathcal{J}(x_0, u) = \|x\|_2^2 + \|u\|_2^2$  (i.e.,  $C = \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $\mathcal{D} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ ,  $J = I$ ), obviously,  $\mathcal{S}_{\text{PT}} \gg 0$  on  $\mathcal{U}_{\text{exp}}(0)$ , equivalently,  $\Sigma$  is positively  $J$ -coercive over  $\mathcal{U}_{\text{exp}}$ , which leads to the existence of a unique optimal control:

**Theorem 4.6 ( $J$ -coercive  $\Rightarrow \exists! J$ -optimal control)** *Assume that  $\Sigma$  is  $J$ -coercive over  $\mathcal{U}_*$ , i.e. that  $\mathcal{S}_{\text{PT}} := \mathcal{D}^* J \mathcal{D} \in \mathcal{B}(\mathcal{U}_*(0), \mathcal{U}_*(0)^*)$  is (boundedly) invertible. Then, for each  $x_0$  s.t.  $\mathcal{U}_*(x_0) \neq \emptyset$ , there exists a unique  $J$ -optimal control.*

*If (f), in addition,  $\mathcal{S}_{\text{PT}} \geq 0$  (or  $\langle \mathcal{D} \cdot, J \mathcal{D} \cdot \rangle \geq 0$ ), i.e.  $\Sigma$  is positively  $J$ -coercive, then the unique  $J$ -optimal control is (strictly) minimizing.*

(The proof is given on p. 57. Note that  $\mathcal{U}_{\text{exp}}(0)$  and  $\mathcal{U}_{\text{out}}(0)$  are Hilbert(izable) spaces, hence  $\mathcal{U}_{\text{exp}}(0)^* = \mathcal{U}_{\text{exp}}(0)$   $\mathcal{U}_{\text{out}}(0)^* = \mathcal{U}_{\text{out}}(0)$ . See Lemma 11.1 for more.)

Thus, under the standard coercivity condition and the *FCC* ( $\mathcal{U}_*(x_0) \neq \emptyset \forall x_0 \in H$ ), there exists a unique optimal control for each initial state  $x_0$ . Also the converse is true if, e.g.,  $\dim U < \infty$  and  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  (see pp. 66 and 35).

Even better, a unique optimal control can be given in WPLS form (Definition 3.2), i.e., as an output of a system:

**Theorem 4.7 ( $\exists! J$ -optimal  $\Rightarrow \exists \Sigma_{\text{opt}}$ )** *Assume that there is a unique  $J$ -optimal control  $u_{\text{opt}}(x_0)$  for each  $x_0 \in H$ . Then  $\mathcal{K}_{\text{opt}} : x_0 \mapsto u_{\text{opt}}(x_0)$  is a control in WPLS form, i.e.,*

$$\Sigma_{\text{opt}} := \left[ \begin{array}{c|c} \mathcal{A}_{\text{opt}} & \mathcal{B}\tau u_{\text{opt}}(x_0) \\ \hline \mathcal{C}_{\text{opt}} & \mathcal{D}u_{\text{opt}}(x_0) \\ \hline \mathcal{K}_{\text{opt}} & u_{\text{opt}}(x_0) \end{array} \right] : x_0 \mapsto \left[ \begin{array}{c|c} \mathcal{A}x_0 + \mathcal{B}\tau u_{\text{opt}}(x_0) & \\ \hline \mathcal{C}x_0 + \mathcal{D}u_{\text{opt}}(x_0) & \\ \hline u_{\text{opt}}(x_0) & \end{array} \right] \quad (33)$$

*is a WPLS (on  $(\{0\}, H, Y \times U)$ ). We call  $\mathcal{P} := \mathcal{C}_{\text{opt}}^* J \mathcal{C}_{\text{opt}}$  the  $J$ -optimal cost operator (or the Riccati operator). It satisfies  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ , and the minimal cost equals  $\mathcal{J}(x_0, u_{\text{opt}}(x_0)) = \langle x_0, \mathcal{P}x_0 \rangle$  for all initial states  $x_0 \in H$ .*

*If  $\mathcal{U}_* \subset \mathcal{U}_{\text{exp}}$ , then  $\Sigma_{\text{opt}}$  is exponentially stable; if  $\mathcal{U}_* \subset \mathcal{U}_{\text{out}}$ , then  $\Sigma_{\text{opt}}$  is output stable.*

(The proof is given on p. 57. We call  $\mathcal{P} := \mathcal{C}_{\text{opt}}^* J \mathcal{C}_{\text{opt}}$  the  $J$ -optimal cost operator (for  $\Sigma, J, \mathcal{U}_*$ ) whenever  $\mathcal{K}_{\text{opt}}$  is a  $J$ -optimal control in WPLS form, even if it were not unique.)

Obviously, the state and first output of  $\Sigma_{\text{opt}}$  with initial state  $x_0$  are those of  $\Sigma$  with initial state  $x_0$  and input  $u_{\text{opt}}(x_0)$ . The  $J$ -optimal control  $u$  also satisfies  $u(t) = (K_{\text{opt}})_w x(t)$  a.e. for certain  $K_{\text{opt}} \in \mathcal{B}(\text{Dom}(A_{\text{opt}}), U)$ , by Lemma 2.4(c). Moreover,  $A_{\text{opt}} = A + BK_{\text{opt}}$ , where  $A_{\text{opt}}, C_{\text{opt}}, K_{\text{opt}}$  are the generators of  $\Sigma_{\text{opt}}$ .

Since  $u(t) = (K_{\text{opt}})_w x(t)$  for a.e.  $t \geq 0$ , the (Yosida extension  $(K_{\text{opt}})_w$  of the) operator  $K_{\text{opt}}$  is a “generalized state-feedback operator” for  $\Sigma$  in certain sense. However, the feedback loop may be ill-posed (this is not the case if  $\mathcal{S}_{\text{PT}} \gg 0$ , by Theorem 5.1, or if the system is sufficiently regular).

If, e.g.,  $B$  is bounded, then  $K = K_{\text{opt}}$  can be computed from (36), and  $\Sigma_{\text{opt}}$  is the left column of the (well-posed) closed-loop system  $\Sigma_{\text{cl}}$  (see (27), p. 12). In Sections 6–9 we explain in detail when  $K_{\text{opt}}$  is well-posed and how the optimal feedback is determined by different AREs and IREs, in principle as in Theorems 1.1 and 1.3.

**Notes for Section 4:** In Chapter 8 of [M02], everything above and much more is presented, the only exception being that in Theorem 4.6 we no longer require  $Z^s$  to be reflexive.

For the case  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$ , Theorems 4.6 and 4.7 are known for the cost function  $\|y\|_2^2 + \|u\|_2^2$  [FLT88] [Z96] ([Z96] seems to be the only unstable optimization result on WPLSs before [M02]) and for a general  $J$ -coercive cost function in the stable case [S98c].

(To be exact, in [S98b] the  $\mathcal{U}_{\text{out}}$  minimization problem for jointly stabilizable and detectable WPLSs was reduced to the stable case. It has previously been very difficult to verify the joint assumption, but now Theorem 5.17 can be used for effectively that purpose. However, thanks to Theorem 5.9(iii) (see Theorem 8.4.5(e)&(g1) of [M02]), now any problems over  $\mathcal{U}_{\text{out}}$  can be reduced to the stable case (use Corollaries 5.16, 5.2 and 5.3 for partial control ( $H^\infty$ ) and/or for  $\mathcal{U}_{\text{exp}}$ ). On the other hand, our theory also provides a direct solution.)

In the stable case (with  $L^2(\mathbb{R}_+; U)$  in place of  $\mathcal{U}_*(x_0)$ ), Definition 4.3 is rather old, and for WPLSs it was first used in [S97].

$J$ -coercivity was defined in [S98c] for stable WPLSs (in [M02] for general ones), but equivalent definitions have been very popular for finite-dimensional or other very special systems, as explained in Section 11. See Chapter 11 of [M02] for applications of indefinite  $\mathcal{S}_{\text{PT}}$  to  $H^\infty$  problems.

The sets  $\mathcal{U}_{\text{out}}$ ,  $\mathcal{U}_{\text{exp}}$  and  $\mathcal{U}_{\text{str}}$  have been used (at least implicitly) for decades, but we have not seen a unified approach before [M02], nor (indefinite unstable versions of) any of the results of this section (not even for finite-dimensional systems). See the notes in Chapter 8 of [M02] for further comments.

## 5 Minimizing control, stabilization and coprime factorizations

In this section, we shall study uniformly positive cost functions ( $\mathcal{S}_{\text{PT}} \gg 0$ ) only. By Theorems 4.6 and 4.7, we already know that in this case the FCC leads to the existence of a unique minimizing control in WPLS form. In Theorem 5.1 we shall show that this control is actually given by a (minimizing, well-posed) state-feedback pair. The remainder of the section consists of corollaries to that theorem: we derive numerous simple but important consequences on stabilization and coprime factorizations. They are the main results of this article along with the Riccati equation theory of Sections 6–7.

As mentioned above, in the uniformly positive case ( $\langle \mathcal{D}u, J\mathcal{D}u \rangle \geq \epsilon \|u\|_{\mathcal{U}_*}^2$  ( $u \in \mathcal{U}_*(0)$ )) with the FCC, the unique optimal control is always given by a (well-posed) state-feedback pair:

**Theorem 5.1** ( $\mathcal{S}_{\text{PT}} \gg 0 \Rightarrow \exists [\mathcal{K} | \mathcal{F}]$ ) *Assume that  $\mathcal{S}_{\text{PT}} \gg 0$  and  $\vartheta = 0$ . Then the FCC is satisfied iff there is a  $J$ -optimal state-feedback pair.*

(The proof is given on p. 62. Recall that  $\vartheta = 0$  when  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  or  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$ .)

By Lemma 4.4(d), here a state-feedback pair is  $J$ -optimal iff it is minimizing. We shall show in Corollary 7.5 that also the existence of a  $\mathcal{U}_*$ -stabilizing solution of the Riccati equation is equivalent to the FCC.

By setting  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  and making the system coercive without affecting  $\mathcal{A}$  or  $\mathcal{B}$ , we can obtain the perfect generalization of a classical finite-dimensional result (and Corollary 1.2), thus solving the well-known open problem:

**Corollary 5.2 (Optimizable  $\Leftrightarrow$  Exp. stabilizable)** *A WPLS is optimizable iff it is exponentially stabilizable.*

The WPLS (or the pair  $[\mathcal{A} | \mathcal{B}]$ ) is called *optimizable* iff  $\mathcal{U}_{\text{exp}}(x_0) \neq \emptyset \forall x_0 \in H$  (i.e., the state-FCC holds). In addition to the corresponding IRE (see Corollary 7.5(c)), one equivalent condition is that a certain (non-integral) Riccati equation has a nonnegative solution, as will be shown in [M03b] (if  $0 \in \rho(A)$ , then the equation becomes  $\mathcal{P}^2 = (A_-^* + \mathcal{P})(I + B_- B_-^*)^{-1}(A_- + \mathcal{P})$ , where  $A_- := A^{-1}$ ,  $B_- := A^{-1}B$  are bounded).



By duality, the corollary implies that a WPLS is estimatable iff it is exponentially detectable. ( $\Sigma$  is called *estimatable* (resp. *exponentially detectable*) iff  $\Sigma^d$  is optimizable (resp. *exponentially stabilizable*) i.e., iff  $(A^* \mid C^*)$  is optimizable.)

As above, by  $(A, B)$ ,  $(A \mid B)$  or  $[\mathcal{A} \mid \mathcal{B}]$  we refer to a system with zero output (to  $[\frac{\mathcal{A}}{0} \mid \frac{\mathcal{B}}{0}]$ ), although the concepts of Corollaries 5.2 and 5.3 are independent of the second row ( $[\mathcal{C} \mid \mathcal{D}]$ ) of the system.

**Proof of Corollary 5.2:** Set  $\mathcal{C} = [\frac{\mathcal{A}}{0}]$ ,  $\mathcal{D} = [\frac{\mathcal{B}^\tau}{I}]$ ,  $J = I$ ,  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$ . Then, by Theorem 5.1, there is a  $J$ -optimal state-feedback pair iff  $[\mathcal{A} \mid \mathcal{B}]$  is optimizable. By, e.g., Theorem 4.7, the pair is exponentially stabilizing (equivalently,  $\mathcal{A}_\circ$  is exponentially stable; recall that  $\Sigma_{\text{opt}} = \Sigma_\circ [\frac{I}{0}]$ ).  $\square$

If  $(A, [B_1 \ B_2])$  is optimizable through the first input, then it is exponentially stabilizable through the first input:

**Corollary 5.3 (( $A, B_1$ ) exp.stab.)** *If  $(A, [B_1 \ B_2])$  is a WPLS and  $(A, B_1)$  is optimizable, then  $(A, [B_1 \ B_2])$  has an exponentially stabilizing state-feedback pair of form  $[\mathcal{K} \mid \mathcal{F}] = [\mathcal{K}_1 \mid \mathcal{F}_{01} \ \mathcal{F}_{02}]$ .*

Note that an arbitrary exponentially stabilizing state-feedback pair  $[\mathcal{K}_1 \mid \mathcal{F}_{11}]$  for  $[\mathcal{A} \mid \mathcal{B}_1]$  need not be extendable for  $[\mathcal{A} \mid \mathcal{B}_1 \ \mathcal{B}_2]$  (i.e.,  $K_1$  and  $B_2$  might be “incompatible”, that is, “ $K_1(\cdot - A)^{-1}B_2$ ” need not be well-posed), by Example 6.6.23 of [M02] (there no pair of form  $[\mathcal{K} \mid \mathcal{F} \ *]$  is admissible for the WPLS  $[\frac{\mathcal{A}}{0} \mid \frac{\mathcal{B}}{0} \ \mathcal{H}]$ , if  $[\mathcal{K} \mid \mathcal{F}]$  and  $[\frac{\mathcal{H}}{\mathcal{G}}]$  are the exponentially stabilizing pairs (“lower row” and “right column”) of the example).

Nevertheless, one can conclude from the proof below (as in the proof of Corollary 5.16) that the  $(\|x\|_2^2 + \|u\|_2^2)$ -minimizing pair  $[\mathcal{K}_1 \mid \mathcal{F}_{11}]$  for  $(A \mid B_1)$  (which is unique modulo (28)) is necessarily admissible with any  $B_2$  s.t.  $(A \mid B_2)$  is a WPLS (even though  $K_1$  depends on  $B_1$ ), i.e., that it satisfies the requirements of the corollary with some  $\mathcal{F}_{12} \in \text{TIC}_\infty(U_2, U_1)$ .

Corollary 5.3 clarifies the assumptions for  $H^\infty$  control problems (see [M02], Chapters 11–12).

**Proof of Corollary 5.3:** Let  $[\mathcal{A} \mid [\mathcal{B}_1 \ \mathcal{B}_2]]$  be a WPLS on  $(U_1 \times U_2, H, -)$  (it might have a second row, but it has no influence on the problem). Define  $\mathcal{C} := [\frac{\mathcal{A}}{0}]$ ,  $\mathcal{D} := [\frac{\mathcal{B}_1^\tau \ \mathcal{B}_2^\tau}{0}]$ ,  $\mathcal{Q} := 0$ ,  $\mathcal{R} := [0 \ I]$ ,  $Z^u := L^2$ ,  $Z^s := \{0\}$ ,  $\vartheta = 0$ . It follows that  $y = [\frac{x}{u}]$  and  $\mathcal{U}_*(x_0) = \{[\frac{u_1}{0}] \in L^2(\mathbb{R}_+; U_1 \times U_2) \mid x \in L^2\}$  (note that Standing Hypothesis 4.1 is satisfied).

The norm  $\|\mathcal{D}u\|_2 = \|\frac{x}{u}\|_2$  is obviously equivalent to  $\|u\|_{\mathcal{U}_*} := \max\{\|u\|_2, \|\mathcal{D}u\|_2, \|0\|\}$ , (which is complete, by Lemma 4.4(a)), hence  $\mathcal{S}_{\text{PT}} \geq \epsilon I$  for some  $\epsilon > 0$  (when we set  $J := I$ ). We have  $\mathcal{U}_*(x_0) \neq \emptyset \ \forall x_0 \in H$ , by the optimizability of  $[\mathcal{A} \mid \mathcal{B}_1]$ . Thus, Theorem 5.1 implies that there is a  $J$ -optimal state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$ .

Set  $\mathcal{X} := I - \mathcal{F} \in \mathcal{GTIC}_\infty(U)$ . Fix  $\alpha$  big enough, so that  $\hat{\mathcal{X}}(\alpha) \in \mathcal{GB}(U)$ . By Lemma 3.7, we can redefine  $[\mathcal{K} \mid \mathcal{F}]$  so that  $\hat{\mathcal{F}}(\alpha) = 0$  (without affecting  $\mathcal{K}_\circ$ ). But  $\mathcal{K}_\circ x_0 \in \mathcal{U}_*(x_0) \ \forall x_0 \in H$  implies that  $(\mathcal{K}_2)_\circ = 0$ . Therefore,  $\mathcal{M} := I + \mathcal{F}_\circ = [\frac{*}{M_{21} \ M_{22}}]$ , where  $M_{21}, M_{22}$  are constants, by, e.g., (171). Since  $\hat{\mathcal{M}}(\alpha) = (I - \hat{\mathcal{F}}(\alpha))^{-1} = [\frac{I \ 0}{0 \ I}]$ , we have  $M_{21} = 0$ ,  $M_{22} = I$ , hence  $\hat{\mathcal{F}} = [\frac{*}{0 \ 0}]$ . But  $\mathcal{K} = \mathcal{M}^{-1} \mathcal{K}_\circ = [\frac{*}{0}]$ , hence  $[\mathcal{K} \mid \mathcal{F}]$  is as required (since  $\mathcal{K}_\circ x_0 \in \mathcal{U}_*(x_0) \subset \mathcal{U}_{\text{exp}}(x_0) \ \forall x_0 \in H$ , the pair  $[\mathcal{K} \mid \mathcal{F}]$  is exponentially stabilizing).  $\square$

Before going on, we need a few concepts on coprimeness and factorization. (Recall from Definition 2.14. that the maps in  $\text{TIC} := \text{TIC}_0$  are called *stable*.)

**Definition 5.4 ([q.]r.c., [q.]r.c.f., d.c.f.)** (a) *We call  $\mathcal{N} \in \text{TIC}(U, Y)$ ,  $\mathcal{M} \in \text{TIC}(U)$  right coprime (r.c.) if  $[\mathcal{X} \ -\mathcal{Y}] [\frac{\mathcal{N}}{\mathcal{M}}] = I$  for some  $\mathcal{X}, \mathcal{Y} \in \text{TIC}$ ; quasi-right coprime (q.r.c.) if  $u \in L^2 \Leftrightarrow [\frac{\mathcal{N}}{\mathcal{M}}] u \in L^2$  whenever  $u \in L^2_\omega(\mathbb{R}_+; U)$ ,  $\omega \in \mathbb{R}$ .*

(b1) *Let  $\mathcal{D} \in \text{TIC}_\infty(U, Y)$ . We call  $\mathcal{N} \mathcal{M}^{-1}$  a right factorization of  $\mathcal{D}$  if  $\mathcal{N}, \mathcal{M} \in \text{TIC}$ ,  $\mathcal{M} \in \mathcal{GTIC}_\infty(U)$  and  $\mathcal{D} = \mathcal{N} \mathcal{M}^{-1}$ . It is called a [quasi-]right-coprime factorization ([q.]r.c.f.) if, in addition,  $\mathcal{N}, \mathcal{M}$  are [q.]r.c.*

(b2) Let  $\mathcal{D} \in \text{TIC}_\infty(U, Y)$ . We call  $\mathcal{N}\mathcal{M}^{-1} = \tilde{\mathcal{M}}^{-1}\tilde{\mathcal{N}}$  a doubly coprime factorization (d.c.f.) of  $\mathcal{D}$  if  $\begin{bmatrix} \mathcal{M} & \mathcal{Y} \\ \mathcal{N} & \mathcal{X} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{\mathcal{X}} & -\tilde{\mathcal{Y}} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix} \in \mathcal{GTIC}(U \times Y)$  for some  $\mathcal{Y}, \mathcal{X}, \tilde{\mathcal{Y}}, \tilde{\mathcal{X}} \in \text{TIC}$ ,  $\mathcal{M} \in \text{TIC}_\infty(U)$  and  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$ .

(c) By the coprimeness of  $\hat{\mathcal{N}}, \hat{\mathcal{M}} : \mathbb{C}^+ \rightarrow \mathcal{B}$  we refer to the coprimeness of  $\mathcal{N}$  and  $\mathcal{M}$  (see Theorem 2.5) etc.

(d) Replace all maps by their adjoints (and  $U$  by  $Y$  and  $Y$  by  $U$ ) to obtain the “left” definitions (e.g., “l.c.”) corresponding to (a) and (b1).

(The minus signs are due to historical reasons. Under (b2), we have  $\tilde{\mathcal{M}} \in \mathcal{GTIC}_\infty(Y)$ , and  $\mathcal{D} = \tilde{\mathcal{M}}^{-1}\tilde{\mathcal{N}}$  is a l.c.f., by Lemma 6.5.9 of [M02].)

We recall some basic properties of coprimeness from [M02]:

**Lemma 5.5 (a1)** If  $\mathcal{N}, \mathcal{M}$  are r.c., then  $\hat{\mathcal{N}}^*\hat{\mathcal{N}} + \hat{\mathcal{M}}^*\hat{\mathcal{M}} \geq \epsilon I$  on  $\mathbb{C}^+$  for some  $\epsilon > 0$ .

If  $\dim U < \infty$ , then also the converse is true, and any r.c. pair  $\mathcal{N}, \mathcal{M}$  can be extended to an invertible element  $\begin{bmatrix} \mathcal{M} & \mathcal{Y} \\ \mathcal{N} & \mathcal{X} \end{bmatrix} \in \mathcal{GTIC}$  (which is a d.c.f. iff  $\mathcal{M} \in \mathcal{GTIC}_\infty$ ).

(a2) If  $\hat{\mathcal{N}}^*\hat{\mathcal{N}} + \hat{\mathcal{M}}^*\hat{\mathcal{M}} \geq \epsilon I$  on  $\mathbb{C}^+$  for some  $\epsilon > 0$ , then  $\mathcal{N}, \mathcal{M}$  are q.r.c.

(b) If  $\mathcal{N}, \mathcal{M}$  are r.c., then they are q.r.c.

(c) If  $\mathcal{N}, \mathcal{M}$  are q.r.c., then  $\|\hat{\mathcal{N}}u_0\|_Y + \|\hat{\mathcal{M}}u_0\|_U > 0$  on  $\mathbb{C}^+$  for all  $u_0 \in U$ , and  $\mathcal{N}^*\mathcal{N} + \mathcal{M}^*\mathcal{M} \gg 0$ .

(d) Let  $\mathcal{N}_0\mathcal{M}_0^{-1}$  be a q.r.c.f. of  $\mathcal{D} \in \text{TIC}_\infty(U, Y)$ . Then all right factorizations  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$  are given by  $\mathcal{N} = \mathcal{N}_0\mathcal{E}$ ,  $\mathcal{M} = \mathcal{M}_0\mathcal{E}$  with  $\mathcal{E} \in \text{TIC}(U) \cap \mathcal{GTIC}_\infty(U)$ . Moreover,  $\mathcal{E} \in \mathcal{GTIC}(U)$  iff  $\mathcal{N}\mathcal{M}^{-1}$  is a q.r.c.f.

In particular, if  $\mathcal{D}$  has a r.c.f., then any q.r.c.f. of  $\mathcal{D}$  is a r.c.f.

(e)  $\mathcal{D}$  has a d.c.f. iff  $\mathcal{D}$  and  $\mathcal{D}^d$  have a r.c.f.

Thus, q.r.c. transfer functions do not have common zeros in  $\mathbb{C}^+$  (in the sense of (c)), nor on the imaginary axis ( $\|\hat{\mathcal{N}}u_0\| + \|\hat{\mathcal{M}}u_0\| \geq \epsilon\|u_0\|$  a.e. on  $i\mathbb{R}$  for all  $u_0 \in U$ , by (c)); see also the comments below Example 5.14. By Theorem 5.21, “and  $\mathcal{D}^d$ ” can be removed from (e). See [M02], Sections 6.4–6.5 for further results and details.

In Corollary 5.13 we will show that any map having a right factorization has a q.r.c.f. By Example 5.14, a q.r.c.f. need not be a r.c.f. (cf. (d)). Nevertheless, any rational q.r.c.f. is a r.c.f. (by Lemma 6.5.3(b) of [M02]); a generalization of this result can be derived from Lemma 5.12. However, not all well-posed maps have a right factorization:

**Example 5.6** There exists  $\mathcal{D} \in \text{TIC}_\infty(\mathbb{C})$  that does not have a right factorization.  $\triangleleft$

Indeed, set  $\hat{\mathcal{D}}(s) := (s-1)^{-1/2}$ , so that  $\mathcal{D} \in \text{TIC}_\omega(\mathbb{C}) \forall \omega > 1$ . Then  $\hat{\mathcal{D}}$  has an essential singularity at  $s = 1$ , whereas maps in  $\text{TIC}_\infty(\mathbb{C})$  having a right factorization have meromorphic extensions to  $\mathbb{C}^+$ , by Corollary 5.10.

By Corollary 5.13 and Theorem 5.21, no realization of the above  $\mathcal{D}$  is output-stabilizable nor dynamically stabilizable. See those corollaries also for equivalent conditions for the existence of [quasi]-coprime factorizations.

Constructive formulas (from the solutions of AREs or IREs) for [q.]r.c.f.’s and d.c.f.’s are given in and below Corollary 7.5. For that purpose one has to use an output-stabilizable realization of  $\mathcal{D}$ ; to get a d.c.f. also the dual condition is required; cf. Theorem 5.9 (or 5.2 with state-FCC) and Theorem 5.17 (or 5.7)). Corresponding formulas also give corresponding stabilizing controllers or state-feedback pairs etc.

**Proof of Lemma 5.5:** (a1) Take  $\epsilon := 1/\|\begin{bmatrix} \mathcal{X} & \mathcal{Y} \end{bmatrix}\|^2$ . The converse follows from the Corona Theorem and the extension from Tolokonnikov’s Lemma, but neither holds when  $\dim U = \infty$ ; see Theorem 4.1.6 and Lemma 6.5.3(b) of [M02] for details (and for similar results for  $\text{MTIC}^{\text{L}^1}$  or other sets in place of  $\text{TIC}$ ).

(a2) This follows from Lemma 4.1.8(g) of [M02].

(b) Now  $u = \begin{bmatrix} \mathcal{X} & \mathcal{Y} \end{bmatrix} \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} u \in \text{L}^2$  when  $\begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} u \in \text{L}^2$ . (Alternative proof: (a1)&(a2).)

(c) Assume that  $\begin{bmatrix} \hat{\mathcal{N}} \\ \hat{\mathcal{M}} \end{bmatrix} (s_0)u_0 = 0$  for some  $s_0 \in \mathbb{C}^+$ ,  $u_0 \in U$ . Set  $\omega := \text{Re } s_0 + 1$ ,  $u(t) := e^{s_0 t} u_0$  (i.e.,  $\hat{u}(s) := (s-s_0)^{-1}u_0$ ). Then  $u \in \text{L}_\omega^2(\mathbb{R}_+; U)$  but  $\begin{bmatrix} \hat{\mathcal{N}} \\ \hat{\mathcal{M}} \end{bmatrix} \hat{u} \in \text{H}^2(\mathbb{C}^+; Y \times U)$  (because it is holomorphic and bounded and  $\leq M/|\text{Im } s|$  for big  $|\text{Im } s|$ ), i.e.,  $\begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} u \in \text{L}^2(\mathbb{R}_+; U \times Y)$ .

(d) This is Lemma 6.4.5(b)&(c) of [M02] (set  $\mathcal{E} := \mathcal{M}_0^{-1}\mathcal{M} \in \mathcal{GTIC}_\infty(U)$  and use [q.]r.c.).

(e) This is Lemma 4.3(iii) of [S98a].  $\square$

As noted below Corollary 5.2, we know that  $\Sigma$  and  $\Sigma^d$  satisfy the state-FCC iff  $\Sigma$  is exponentially stabilizable and exponentially detectable; in fact, then it is exponentially *jointly* stabilizable and detectable (the terminology will be explained below the corollary):

**Corollary 5.7** ( $\mathcal{U}_{\text{exp}} \neq \emptyset \neq \mathcal{U}_{\text{exp}}^{\Sigma^d} \Leftrightarrow$  **jointly stab.&det.**) *The following are equivalent:*

- (i)  $\Sigma$  is exponentially jointly stabilizable and detectable.
- (ii)  $\Sigma$  and  $\Sigma^d$  satisfy the state-FCC.
- (iii)  $\Sigma$  satisfies the output-FCC and  $\Sigma^d$  the state-FCC.
- (iv)  $\Sigma$  satisfies the state-FCC and  $\Sigma^d$  the output-FCC.
- (v) There is an exponentially stabilizing dynamic feedback controller for  $\Sigma$  with internal loop.

Moreover, any output-stabilizing state-feedback pair for an estimatable system is exponentially r.c.-stabilizing. Any exponentially jointly stabilizing pairs for  $\Sigma$  define (through (35)) an exponential doubly coprime factorization of the I/O map  $\mathcal{D}$  of  $\Sigma$ .

As before, output-FCC (FCC for  $\mathcal{U}_{\text{out}}$ ) means that  $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset \forall x_0 \in H$ . Condition (i) means that  $\Sigma$  can be extended to a WPLS

$$\Sigma_{\text{Joint}} := \left[ \begin{array}{c|cc} \mathcal{A} & \mathcal{H} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{G} & \mathcal{D} \\ \hline \mathcal{K} & \mathcal{E} & \mathcal{F} \end{array} \right] \quad (34)$$

(on  $(Y \times U, H, Y \times U)$ ) s.t.  $(\Sigma_{\text{Joint}})_L$  and  $(\Sigma_{\text{Joint}})_{\tilde{L}}$  are exponentially stable, where  $L = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$  and  $\tilde{L} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$  (see Lemma 3.1). This says that  $\Sigma_{\text{Joint}}$  becomes exponentially stable when the added output is connected to the original input or the original output is connected to the added input. (*Jointly stabilizable and detectable* means the same except that  $(\Sigma_{\text{Joint}})_L$  and  $(\Sigma_{\text{Joint}})_{\tilde{L}}$  need be merely stable.)

By Lemma 2.2, the two closed-loop systems are exponentially stable iff  $\mathcal{A}_L = \mathcal{A} + \mathcal{B}\tau(I - \mathcal{F})^{-1}\mathcal{K}$  and  $\mathcal{A}_{\tilde{L}} = \mathcal{A} + \mathcal{H}\tau(I - \mathcal{G})^{-1}\mathcal{C}$  are exponentially stable (i.e., iff they map  $H$  into  $L^2(\mathbb{R}_+; H)$ ). In this case, we call  $[\mathcal{K} | \mathcal{F}]$  and  $[\frac{\mathcal{K}}{\mathcal{G}}]$  *exponentially jointly stabilizing pairs* for  $\Sigma$ . It follows that  $[\mathcal{K} | \mathcal{F}]$  is (an) exponentially stabilizing (state-feedback pair) and  $[\frac{\mathcal{K}}{\mathcal{G}}]$  is (an) *exponentially detecting (output injection pair)* for  $\Sigma$  (see Section 6.6 of [M02] for further explanations and results).

Under (i), the  $\mathcal{GTIC}(U \times Y)$  maps

$$\begin{bmatrix} \mathcal{M} & \mathcal{Y}_1 \\ \mathcal{N} & \mathcal{X}_1 \end{bmatrix} := \begin{bmatrix} I + \mathcal{F}_L & -\mathcal{E}_L \\ \mathcal{D}_L & I - \mathcal{G}_L \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\mathcal{X}} & -\tilde{\mathcal{Y}} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix} := \begin{bmatrix} I - \mathcal{F}_{\tilde{L}} & \mathcal{E}_{\tilde{L}} \\ -\mathcal{D}_{\tilde{L}} & I + \mathcal{G}_{\tilde{L}} \end{bmatrix} \quad (35)$$

are the inverses of each other (by a direct computation, see Theorem 4.4 of [S98a] for the details; actually these maps are the inverses of each other even when they are unstable, it suffices that  $L, \tilde{L}$  are admissible). It follows that (35) defines a d.c.f. of  $\mathcal{D}$  (actually, an *exponential d.c.f.*, which means that (35)  $\in \mathcal{GTIC}_{\omega}(U \times Y)$  for some  $\omega < 0$ ).

As noted below Corollary 5.3, arbitrary exponentially stabilizing and detecting pairs  $[\mathcal{K} | \mathcal{F}]$  and  $[\frac{\mathcal{K}}{\mathcal{G}}]$  for  $\Sigma$  need not be jointly admissible for  $\Sigma$  (i.e., no  $\mathcal{E}$  makes (34) a WPLS). However, the  $\|x\|_2^2 + \|u\|_2^2$ -minimizing pair  $[\mathcal{K} | \mathcal{F}]$  is jointly admissible with any admissible  $[\frac{\mathcal{K}}{\mathcal{G}}]$ , as noted in 2<sup>o</sup> below. By duality, any admissible  $[\mathcal{K} | \mathcal{F}]$  is jointly admissible with certain exponentially stabilizing  $[\frac{\mathcal{K}}{\mathcal{G}}]$  (if any exists, i.e., if  $\Sigma^d$  satisfies the state-FCC).

The “moreover” claim means that if  $\Sigma$  is estimatable and an admissible state-feedback pair  $[\mathcal{K} | \mathcal{F}]$  makes  $\mathcal{C}_{\circ}$  and  $\mathcal{K}_{\circ}$  stable, then it actually makes  $\Sigma_{\circ}$  exponentially stable and  $\mathcal{N}, \mathcal{M}$  exponentially r.c.<sup>5</sup>

<sup>5</sup>The maps  $\mathcal{N}, \mathcal{M}$  are called *exponentially r.c.* if there exist  $\omega < 0$ ,  $[\tilde{\mathcal{Y}} | \tilde{\mathcal{X}}] \in \text{TIC}_{\omega}(Y \times U, U)$  s.t.  $\mathcal{N}, \mathcal{M} \in \text{TIC}_{\omega}$  and  $[-\tilde{\mathcal{Y}} | \tilde{\mathcal{X}}] \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} = I$ . Recall that  $\mathcal{M} := (I - \mathcal{F})^{-1}$ ,  $\mathcal{N} := \mathcal{D} = \mathcal{D}_{\circ}$ , so that  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$ .

Condition (v) means roughly a system in place of  $L$  in Figure 2 (p. 10) s.t. the connection stabilizes both systems exponentially. It is further explained in Section 7.2 of [M02]. By Theorem 5.21, “with internal loop” may be removed if  $\dim U, \dim Y < \infty$  (take any jointly exponentially stabilizable and detectable realization of any  $\mathcal{T}$  s.t.  $\begin{bmatrix} I & -\mathcal{T} \\ -\mathcal{D} & I \end{bmatrix}^{-1}$  is exponentially stable; cf. Theorem 7.2.3(d)&(c1) of [M02]), but the general case is open. **Proof of Corollary 5.7:** The last claim was shown above (actually, in Theorem 4.4 of [S98a]). By the dual of  $2^\circ$ , any output-stabilizing state-feedback pair for  $\Sigma$  is exponentially jointly (coprime) stabilizing with some  $\begin{bmatrix} \mathcal{K} \\ \mathcal{G} \end{bmatrix}$  and (interaction operator)  $\mathcal{E}$ . This proves the “moreover” claim, hence only the equivalence remains to be proved.

$1^\circ (i) \Rightarrow (iii) \Rightarrow (ii)$ : The first implication is obvious. Assume then (iii), so that there exists  $\begin{bmatrix} \mathcal{K} \mid \mathcal{F} \end{bmatrix}$  s.t.  $\mathcal{C}_\circ, \mathcal{K}_\circ$  are stable and  $\Sigma$  is estimatable ( $\mathcal{U}_{\text{exp}}^{\Sigma^d}(x_0) \neq \emptyset \forall x_0 \in H$ ), hence so is  $\Sigma_{\text{ext}}$ , hence  $\Sigma_\circ$ , hence  $\Sigma_\circ$  is exponentially stable, by Theorem 7.3 and Proposition 6.2 of [WR00].

$2^\circ (ii) \Rightarrow (i)$ : By the dual of Corollary 5.2,  $\Sigma$  can be extended to a WPLS  $\Sigma_2 := \begin{bmatrix} \mathcal{A} \mid \mathcal{B} \mid \mathcal{K} \\ \mathcal{C} \mid \mathcal{D} \mid \mathcal{G} \end{bmatrix}$  that becomes exponentially stable under the static output feedback through  $L = \begin{bmatrix} 0 \\ I \end{bmatrix}$  (i.e., under input  $u = u_L + Ly$ , where  $u_L : \mathbb{R}_+ \rightarrow U \times Y$  is an external input and  $y = \mathcal{C}x_0 + \begin{bmatrix} \mathcal{D} & \mathcal{G} \end{bmatrix} u$ ; see Definition 6.6.21 of [M02] or Lemma 3.1 for details).

By Corollary 5.3, there is an exponentially stabilizing state-feedback pair  $\begin{bmatrix} \mathcal{K} \mid \mathcal{F} \mid \mathcal{G} \end{bmatrix}$  for  $\Sigma_2$  (the  $\|x\|_2^2 + \|u\|_2^2$ -minimizing one, as noted below Corollary 5.3). Obviously, (34) is a WPLS and  $\mathcal{A}_L$  and  $\mathcal{A}_{\tilde{L}}$  are exponentially stable.

$3^\circ (i)-(iv)$  are equivalent: By  $1^\circ-2^\circ$ , (i)–(iii) are equivalent. But (iv) is exactly (iii) applied to  $\Sigma^d$ .

$4^\circ (i) \Rightarrow (v) \Rightarrow (ii)$ : This was given in Theorem 7.2.4(b)&(a) of [M02]. (See [M02] for the definition and further results and notes.)  $\square$

We also conclude the equivalence of the weak and strong forms of the standard assumption for the  $H^\infty$  Four-Block Problem (“stabilizable through  $u_1$  and detectable through  $y_2$ ”):

**Remark 5.8** Assume that  $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$  and  $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ . Then there are exponentially jointly stabilizing and detecting pairs through  $B_1$  and  $C_2$  (as in (12.76)–(12.77) of [M02]) iff  $(A, B_1)$  is exponentially stabilizable and  $(A, C_2)$  is exponentially detectable

By Corollary 5.2, a third equivalent condition is that  $(A, B_1)$  and  $(A^*, C_2^*)$  satisfy the state-FCC. From the proof of Lemma 12.5.4 of [M02] we observe that Hypothesis 12.5.1 is exponentially satisfied iff the above conditions hold and  $\mathcal{D}_{11}$  and  $\mathcal{D}_{22}^d$  are  $I$ -coercive (over  $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{out}}$ ).

To obtain similar results on non-exponentially stabilizing  $H^\infty$  controllers, one should use Corollary 5.16 and work as in the proof of Theorem 5.17.

**Proof of Remark 5.8:** Choose first  $\begin{bmatrix} \mathcal{K} \mid \mathcal{F} \end{bmatrix}$  as in Corollary 5.3. Then work as in  $2^\circ$  of the proof of Corollary 5.7 but choose the (permuted dual  $\begin{bmatrix} \mathcal{K}_2^d \mid \mathcal{G}_{12}^d & \mathcal{G}_{22}^d \end{bmatrix}$  of the) pair  $\begin{bmatrix} \mathcal{K} \\ \mathcal{G} \end{bmatrix}$  as in Corollary 5.3, so that its first column is zero.  $\square$

Using Theorem 5.1, we can deduce that the output-FCC implies the existence of an output-stabilizing state-feedback pair for  $\Sigma$  (namely the  $\|u\|_2^2 + \|y\|_2^2$ -minimizing one), thus generalizing Corollary 1.4. Actually, we can show that this specific pair is *SOS-stabilizing* (which means that  $\mathcal{C}_\circ, \mathcal{D}_\circ, \mathcal{K}_\circ, \mathcal{F}_\circ$  are stable, i.e., that they map  $H$  or  $L^2$  into  $L^2$ ) and leads to a (normalizable) quasi-right coprime factorization of  $\mathcal{D}$ :

**Theorem 5.9** ( $\mathcal{U}_{\text{out}}$ : FCC  $\Leftrightarrow \exists \begin{bmatrix} \mathcal{K} \mid \mathcal{F} \end{bmatrix}$ ) The following are equivalent:

- (i)  $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset \forall x_0 \in H$ .
- (ii) There is an output-stabilizing state-feedback pair  $\begin{bmatrix} \mathcal{K} \mid \mathcal{F} \end{bmatrix}$  for  $\Sigma$ .
- (iii) There is a SOS-stabilizing state-feedback pair  $\begin{bmatrix} \mathcal{K} \mid \mathcal{F} \end{bmatrix}$  for  $\Sigma$  s.t.  $\mathcal{D} = \mathcal{N} \mathcal{M}^{-1}$  is a q.r.c.f. and  $\mathcal{N}^* \mathcal{N} + \mathcal{M}^* \mathcal{M} = I$ .

(The proof is given on p. 62.)

Conversely, any map having a q.r.c.f. (equivalently, a right factorization) has a realization satisfying (i)–(iii), by Corollary 5.13(i). Recall that the output-FCC (i) means that for all  $x_0 \in H$ , there exists  $u \in L^2(\mathbb{R}_+; U)$  s.t.  $y \in L^2$ .

A q.r.c.f. is unique modulo an element of  $\mathcal{GTIC}(U)$ . If(f)  $\mathcal{D}$  has a right-coprime factorization, then any q.r.c. factorization of  $\mathcal{D}$  is right-coprime. See Lemma 6.4.5(c) of [M02] for proofs.

By Lemma 12.2(c), we have  $\begin{bmatrix} \mathcal{D} \\ I \end{bmatrix} \mathcal{U}_{\text{out}}(0) = \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} L^2(\mathbb{R}_+; U)$ .

The maps  $\mathcal{N}, \mathcal{M}$  are actually r.c. in (iii) if, e.g.,  $\dim U < \infty$  and  $\sigma(A)$  is nice (see Lemma 5.12 below). Any q.r.c.f. can be “normalized” to satisfy  $\mathcal{N}^* \mathcal{N} + \mathcal{M}^* \mathcal{M} = I$ , by Lemma 5.5(c)&(d) and Theorem 5.26(a).

If the input space is finite-dimensional, then the FCC implies that  $\mathcal{D}$  is *meromorphic* (i.e., for any  $s_0 \in \mathbb{C}^+$ , there is  $n \in \mathbb{N}$  s.t.  $s \mapsto (s - s_0)^n \hat{\mathcal{D}}(s)$  is holomorphic on a neighborhood of  $s_0$ ):

**Corollary 5.10 ( $\hat{\mathcal{D}}$  is meromorphic)** *Assume that  $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset \forall x_0 \in H$  and  $\dim U < \infty$ . Then  $\hat{\mathcal{C}}, \hat{\mathcal{D}}$  are meromorphic on  $\mathbb{C}^+$  (and so are  $\hat{\mathcal{K}}, \hat{\mathcal{F}}, \hat{\mathcal{M}}^{-1}$  for any output-stabilizing  $[\mathcal{K} | \mathcal{F}]$ ).*

*If  $\mathcal{U}_{\text{exp}}(x_0) \neq \emptyset \forall x_0 \in H$ , then  $\hat{\mathcal{A}}, \widehat{\mathcal{B}\tau}, \hat{\mathcal{C}}, \hat{\mathcal{D}}$  are meromorphic on  $\mathbb{C}_{-\delta}^+$  for some  $\delta > 0$  (and so are  $\hat{\mathcal{K}}, \hat{\mathcal{F}}, \hat{\mathcal{M}}^{-1}$  for any exponentially stabilizing  $[\mathcal{K} | \mathcal{F}]$ ).*

In particular,  $\hat{\mathcal{D}} = \hat{\mathcal{D}}_{\Sigma}$  and  $\hat{\mathcal{C}} = C(\cdot - A)^{-1}$  a.e. on  $\mathbb{C}^+$  (or on  $\mathbb{C}_{-\delta}^+$ ), by Lemma A.2(f)&(a)&(b1) (which defines the above symbols).

If we would define “meromorphic” as quotient of analytic maps, assumption  $\dim U < \infty$  would be redundant (by the proof below) but now it is not the case, by Example 5.11.

**Proof:** 1° We obtain  $\hat{\mathcal{D}} = \hat{\mathcal{N}} \hat{\mathcal{X}}, \hat{\mathcal{C}} = \widehat{\mathcal{C}_{\circ}} - \hat{\mathcal{D}} \widehat{\mathcal{K}_{\circ}}, \hat{\mathcal{K}} = \hat{\mathcal{X}} \widehat{\mathcal{K}_{\circ}}, \hat{\mathcal{F}} = I - \hat{\mathcal{X}}$  on some right half-plane from Theorem 5.9. Since  $f(s) := \det \hat{\mathcal{M}} \neq 0$ , the only singularities of  $f^{-1}$  (and of  $\hat{\mathcal{X}} = \hat{\mathcal{M}}^{-1}$  and of  $\hat{\mathcal{D}} = \hat{\mathcal{N}} \hat{\mathcal{X}}$ ) on  $\mathbb{C}^+$  are isolated poles (cf. p. 112 of [M02]). It follows that also  $\hat{\mathcal{D}}$  and  $\hat{\mathcal{C}}$  have meromorphic extensions to  $\mathbb{C}^+$  (cf. Remark A.4).

2° If  $[\mathcal{K} | \mathcal{F}]$  is exponentially stabilizing, i.e.,  $-\delta := \omega_{A_{\circ}} < 0$ , then  $\widehat{\Sigma}_{\circ}$  is holomorphic and  $\hat{\mathcal{X}}$  is meromorphic on  $\mathbb{C}_{-\delta}^+$ , hence  $\hat{\mathcal{D}}, \hat{\mathcal{C}}, \hat{\mathcal{K}}$  have meromorphic extensions to  $\mathbb{C}_{-\delta}^+$ , as above, and so do  $\hat{\mathcal{B}} = \widehat{\mathcal{B}_{\circ}} \hat{\mathcal{X}}, \hat{\mathcal{A}} = \widehat{\mathcal{A}_{\circ}} - \hat{\mathcal{B}} \widehat{\mathcal{K}_{\circ}}$ .  $\square$

As mentioned above, the poles of  $\hat{\mathcal{D}}$  need not be isolated when  $\dim U = \infty$ , not even when  $\mathcal{U}_{\text{exp}}(x_0) \neq \emptyset \forall x_0 \in H$ :

**Example 5.11 ( $\hat{\mathcal{D}}$  is not meromorphic).** Let  $\dim U = \infty$ . Then there is an exponentially (r.c.-)stabilizable WPLS  $\Sigma = \begin{bmatrix} \mathcal{A} + \mathcal{B} \\ \mathcal{C} + \mathcal{D} \end{bmatrix}$  s.t.  $\mathcal{D} \in \text{TIC}_{\infty}(U)$  and  $\mathcal{D} = I \mathcal{M}^{-1}$  is an exponential r.c.f. (hence q.r.c.f.) but all points of  $\{z \in \mathbb{C} \mid |z - 5| < 1\}$  are poles of  $\hat{\mathcal{D}}$ .  $\triangleleft$

Note also that  $\mathcal{D}$  also has a normalized exponential r.c.f.  $\mathcal{D} = \tilde{\mathcal{N}} \tilde{\mathcal{M}}^{-1}$  (by Lemma 6.4.7(a)&(c), for some  $\delta > 0$  there is  $\mathcal{X} \in \mathcal{GTIC}_{-\delta}(U)$  s.t.  $\mathcal{X}^* \mathcal{X} = I^* I + \mathcal{M}^* \mathcal{M}$ ; set  $\tilde{\mathcal{N}} := \mathcal{X}^{-1}, \tilde{\mathcal{M}} := \mathcal{M} \mathcal{X}^{-1}$ ).

**Proof:** Take  $s_0 = -1$  and choose an infinite compact  $K \subset \mathbb{C}^+$  (e.g.,  $K = \{z \in \mathbb{C} \mid |z - 5| < 1\}$ ) to obtain, from Lemma 3.3.9 of [M02], a function  $\hat{\mathcal{M}} \in H^{\infty}(\mathbb{C}^+; \mathcal{B}(U))$  (actually,  $\hat{\mathcal{M}} \in H^{\infty}(\mathbb{C}_{-\delta}^+; \mathcal{B}(U))$  for any  $\delta < 1$ ) s.t.  $\mathcal{M} \in \text{TIC} \cap \mathcal{GTIC}_{\infty}(U)$  but  $\hat{\mathcal{D}} := \hat{\mathcal{M}}^{-1}$  has an infinite number of poles (all points of  $K$ ) on  $\mathbb{C}^+$ .

By Corollary 5.13(iii), we already have a “counter-example” to Corollary 5.10, but to make it even more striking, we use a shifted version of Lemma 6.6.29 of [M02] (i.e., we take an exponentially stable realization “ $\Sigma_{\circ}$ ” of  $\begin{bmatrix} \mathcal{N} \\ \mathcal{M} - I \end{bmatrix}$  and apply static feedback  $L := \begin{bmatrix} 0 & I \end{bmatrix}$  to open it, thus obtaining a realization  $\Sigma_{\text{ext}}$  of  $\begin{bmatrix} \mathcal{D} \\ I - \mathcal{D} \end{bmatrix}$ ; then we drop the bottom row (which is an exponentially r.c.-stabilizing state-feedback pair for  $\Sigma$ )) to obtain an exponentially (r.c.-)stabilizable realization of  $\mathcal{D}$ .

“R.c.-” means that  $\mathcal{N} := \mathcal{D} \mathcal{M}$  and  $\mathcal{M}$  are r.c. in Definition 3.5, i.e., that  $\mathcal{N}, \mathcal{M} \in \text{TIC}$  are s.t.  $\tilde{\mathcal{X}} \mathcal{M} - \tilde{\mathcal{Y}} \mathcal{N} = I$  for some  $\tilde{\mathcal{X}}, \tilde{\mathcal{Y}} \in \text{TIC}$ ; “exponential” means that this holds with  $\text{TIC}_{-\delta}$  in place of  $\text{TIC}$  for some  $\delta > 0$ . See Definition 6.6.10 of [M02] for more.)  $\square$

Any  $[\mathcal{K} | \mathcal{F}]$  making  $\mathcal{N}, \mathcal{M}$  q.r.c. actually makes them r.c. if  $\sigma(A)$  is nice and  $\dim U < \infty$ :

**Lemma 5.12 (Nice A: q.r.c.  $\Leftrightarrow$  r.c.)** Assume that  $[\mathcal{X} \mid \mathcal{F}]$ ,  $A$  and  $U$  are as in Lemma A.8 and that  $\hat{\mathcal{M}}(s)$  converges as  $s \in \mathbb{C}^+$ ,  $|s| \rightarrow \infty$ . Then  $\mathcal{N}, \mathcal{M}$  are q.r.c. iff they are r.c. Moreover, there is a rational  $g \in H^\infty(\mathbb{C}^+; \mathbb{C})$  s.t.  $gI_U$  and  $g\hat{\mathcal{D}}$  form a r.c.f.

Recall from Theorem 4.1.6(d) of [M02] that any r.c.f. can be extended to a d.c.f. when  $\dim U < \infty$ .

**Proof:** 1° *R.c.f.:* Now  $M = \lim_{s \rightarrow +\infty} \hat{\mathcal{M}}(s) \in \mathcal{GB}(U)$  (since  $\hat{\mathcal{M}}(s)^{-1} = \hat{\mathcal{X}}(s)$  is uniformly bounded for big  $s$ ). Choose  $\epsilon_K > 0$  s.t.  $M^*M > 2\epsilon_K^2 I$  and a compact  $K \subset \overline{\mathbb{C}^+}$  s.t.  $\hat{\mathcal{M}}^* \hat{\mathcal{M}} \geq \epsilon_K^2$  on  $\overline{\mathbb{C}^+} \setminus K$ .

Set  $\mathcal{E} := [\mathcal{N} \mid \mathcal{M}]$ . By Lemma 5.5(c),  $f(s) := \min_{\|u_0\|_U=1} \|\hat{\mathcal{E}}(s)u_0\| > 0 \ \forall s \in \mathbb{C}^+$  and there is  $\epsilon \in (0, \epsilon_K)$  s.t.  $\hat{\mathcal{E}}^* \hat{\mathcal{E}} \geq 2\epsilon^2$  a.e. on  $i\mathbb{R}$ , hence everywhere on  $i\mathbb{R}$  (use the extensions of Lemma A.8). We conclude that  $\epsilon_1 := \inf_{s \in K} f(s) > 0$ . Therefore,  $f(s) \geq \epsilon_2 := \min\{\epsilon_1, \epsilon_K\} \ \forall s \in \overline{\mathbb{C}^+}$ . By Lemma 5.5(a1),  $\mathcal{N}, \mathcal{M}$  are r.c.

2° *g:* The poles of  $\hat{\mathcal{D}}$  on  $\overline{\mathbb{C}^+}$  are on  $K$ , hence their number is finite; denote them by  $s_1, \dots, s_n$ . Set  $g := \prod_{k=1}^n (s - s_k)/(s + s_k + 1)$  to have  $\|g\|_\infty \leq 1$ . Then  $g\hat{\mathcal{D}} \in H^\infty$  and  $\det(g\hat{\mathcal{D}})$  has no common zeros with  $g$  on  $\overline{\mathbb{C}^+}$ , hence  $g\hat{\mathcal{D}}$  and  $gI_U$  are r.c. (as in 1°).  $\square$

If an I/O map can be written as the quotient of two stable maps, then these stable maps can be chosen to be quasi-right coprime and normalized:

**Corollary 5.13 (q.r.c.f.)** Let  $\mathcal{D} \in \text{TIC}_\infty(U, Y)$ . Then following are equivalent:

- (i)  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$ , where  $\mathcal{N}, \mathcal{M} \in \text{TIC}$ ,  $\mathcal{M} \in \mathcal{GTIC}_\infty(U)$ .
- (ii)  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$ , where  $\mathcal{N}, \mathcal{M}$  are q.r.c.,  $\mathcal{M} \in \mathcal{GTIC}_\infty(U)$ , and  $\mathcal{N}^* \mathcal{N} + \mathcal{M}^* \mathcal{M} = I$ .
- (iii) There is a realization of  $\mathcal{D}$  s.t.  $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset \ \forall x_0 \in H$ .
- (iv) There is a stabilizable realization of  $\mathcal{D}$ .
- (v) For some  $\omega \in \mathbb{R}$  and any  $v \in L_\omega^2(\mathbb{R}_-; U)$ ,  $\mathcal{D} \in \text{TIC}_\omega$  and there exists  $u \in L^2(\mathbb{R}_+; U)$  s.t.  $\pi_+ \mathcal{D}(v + u) \in L^2$ .

Assume (ii). Then all solutions of (ii) are given by  $\begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} = \begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix} E$  ( $E \in \mathcal{GB}(U)$ ,  $E^*E = I$ ).

Assume that (ii) holds and  $\dim U < \infty$ . Then also  $\mathcal{D}^d$  satisfies (ii) (i.e.,  $\mathcal{D}$  has both a q.r.c.f. and a q.l.c.f.). If  $\mathcal{N}, \mathcal{M} \in \text{MTIC}^{L^1}$ , then  $\mathcal{N}, \mathcal{M}$  are actually r.c. and can hence be extended to a d.c.f. in  $\text{MTIC}^{L^1}$ .

See Theorem 5.9 for further equivalent conditions. Condition (iii) (hence (i)–(v)) holds iff  $\mathcal{D}$  is the I/O map of some system having output-stabilizing inputs — in the negative case no reasonable control problems for  $\mathcal{D}$  have solutions.

Condition (v) says that the range of the Hankel operator  $\pi_+ \mathcal{D} \pi_-$  (restricted to some  $L_\omega^2$ ) is contained in the sum of  $L^2$  and the range of the Toeplitz operator  $\pi_+ \mathcal{D} \pi_+$ . If (v) holds for some  $\omega \in \mathbb{R}$ , then it holds for any  $\omega' > \omega$  (because  $L_{\omega'}^2(\mathbb{R}_-; U) \subset L_\omega^2(\mathbb{R}_-; U) \subset$ ).

The function  $\hat{\mathcal{D}}(s) := (s - 1)^{-1/2}$  satisfies  $\mathcal{D} \in \text{TIC}_\omega(\mathbb{C})$  for any  $\omega > 1$ , by Theorem 2.5, but does not satisfy any of (i)–(v), as noted below Example 5.6.

The corollary can be applied to any quadratic minimization problems of even more general systems than WPLSs as long as the stabilizability assumption (v) is satisfied.

**Proof of Corollary 5.13:** 1°  $(ii) \Rightarrow (i)$ : By the definition of “q.r.c.”,  $\mathcal{N}, \mathcal{M} \in \text{TIC}$ .

2°  $(i) \Rightarrow (v)$ : Let  $\omega \geq 0$  be s.t.  $\mathcal{M} \in \mathcal{GTIC}_\omega$ . Since  $\tilde{v} := \pi_- \mathcal{M}^{-1} v \in L_\omega^2(\mathbb{R}_-; U) \subset L^2$ , we have  $u, y \in L^2$ , where  $u := \pi_+ \mathcal{M} \tilde{v}$ ,  $y := \pi_+ \mathcal{N} \tilde{v}$ . But  $v = \pi_- \mathcal{M} \mathcal{M}^{-1} v = \pi_- \mathcal{M} \tilde{v}$ , hence  $\pi_+ \mathcal{D}(v + u) = \pi_+ \mathcal{D}(\pi_- \mathcal{M} \tilde{v} + \pi_+ \mathcal{M} \tilde{v}) = \pi_+ \mathcal{D} \mathcal{M} \tilde{v} = y$ , so (v) holds.

3°  $(v) \Rightarrow (iii)$ : Condition (v) (without  $\mathcal{D} \in \text{TIC}_\omega$ ) is exactly condition (iii) for the  $\omega$ -stable exactly reachable realization  $\left[ \begin{smallmatrix} \tau \pi_- \\ \pi_+ \mathcal{D} \pi_- \end{smallmatrix} \middle| \begin{smallmatrix} \pi_- \\ \mathcal{D} \end{smallmatrix} \right]$  on  $(U, L_\omega^2(\mathbb{R}_-; U), Y)$  (which is a WPLS when  $\mathcal{D} \in \text{TIC}_\omega$ ).

4°  $(iii) \Rightarrow (ii)$ : This follows from Theorem 5.9(iii).

5°  $(ii) \Rightarrow (iv) \Rightarrow (iii)$ : The latter implication is trivial, and the former is from Lemma 6.6.29 of [M02] (in fact, the realization is strongly q.r.c.-stabilizable).

6° *All solutions formula:* It is from Lemma 6.4.5(e) of [M02].

7° *Case  $\dim U < \infty$ :* Choose  $f$  as in the proof of Corollary 5.10, so that  $\hat{\mathcal{X}} = f^{-1}F$  for some  $F \in H^\infty$ . But  $\det \hat{\mathcal{X}} = (\det \hat{\mathcal{M}})^{-1} = f^{-1}$  a.e., hence  $\det F = 1$  on  $\mathbb{C}^+$ , hence

$F^{-1} \in H^\infty$ . Thus,  $fI_U = \hat{\mathcal{M}}F^{-1} = F^{-1}\hat{\mathcal{M}}$  and  $G := f\hat{\mathcal{D}} = \hat{\mathcal{N}}F^{-1}$  are q.r.c., and  $fI_Y$  and  $G$  form a left factorization of  $\hat{\mathcal{D}}$ , since  $f \in \mathcal{GH}_\omega^\infty$  for some  $\omega > 0$ . Thus, (i) (hence (ii)–(v) too) holds for  $\mathcal{D}^d$  too.

8° *R.c.*: By the proof of Lemma 5.12,  $\mathcal{N}, \mathcal{M}$  are r.c. when  $\hat{\mathcal{N}}, \hat{\mathcal{M}}$  are q.r.c. and continuous on  $\overline{\mathbb{C}^+} \cup \{\infty\}$  (the latter holds if  $\mathcal{N}, \mathcal{M} \in \text{MTIC}^{L^1}$ , see p. 29),  $\mathcal{M} \in \mathcal{GTIC}_\infty(U)$  and  $\dim U < \infty$ . By Theorem 4.1.6(d) of [M02], there is an extension (a d.c.f.)  $[\frac{\mathcal{M}}{\mathcal{N}}]_* \in \text{GMTIC}^{L^1}(U \times Y)$ .  $\square$

However, a q.r.c.f. need not be a r.c.f., hence “q.” is not redundant in Theorem 5.9 nor in Corollary 5.13:

**Example 5.14** (q.r.c.f.  $\not\Rightarrow$  r.c.f.). Let  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{N}}$  be the Blaschke products with zeros  $\{n^{-2} \mid n = 2, 3, \dots\}$  and  $\{(n^2 + 1)^{-1} \mid n = 2, 3, \dots\}$ , respectively. Then  $\mathcal{N}\mathcal{M}^{-1}$  is a q.r.c.f. and  $\mathcal{N}^*\mathcal{N} + \mathcal{M}^*\mathcal{M} = 2$  (multiply  $\mathcal{N}$  and  $\mathcal{M}$  by  $2^{-1/2}$  to normalize them as in Corollary 5.13(ii)), but  $\mathcal{N}, \mathcal{M}$  are not r.c., hence  $\mathcal{N}\mathcal{M}^{-1}$  does not have a r.c.f., by Lemma 6.4.5(c) of [M02].  $\triangleleft$

In particular, there is no d.c.f. although  $\mathcal{N}\mathcal{M}^{-1} = \mathcal{M}^{-1}\mathcal{N}$  are a q.r.c.f. and a q.l.c.f. **Proof:** 1°  $\mathcal{M}$  and  $\mathcal{N}$  are q.r.c.: If  $\hat{\mathcal{M}}f, \hat{\mathcal{N}}f \in H^2$ , then  $f$  cannot have singularities on  $\mathbb{C}^+$  (since  $\hat{\mathcal{M}}$  and  $\hat{\mathcal{N}}$  have no common zeros). Thus, the zeros of  $\hat{\mathcal{M}}f$  equal those of  $f$  combined with those of  $\hat{\mathcal{M}}$ , hence  $B_{\hat{\mathcal{M}}f} = B_{\hat{\mathcal{M}}}B_f$  (where  $B_f$  is the Blaschke product formed with the zeros of  $f$  etc.). But  $\hat{\mathcal{M}} = B_{\hat{\mathcal{M}}}$ , hence  $H^2 \ni \hat{\mathcal{M}}f/B_{\hat{\mathcal{M}}f} = f/B_f$ , by pp. 132–133 of [H62], hence  $H^2 \ni B_f \cdot f/B_f = f$ .

2° One easily verifies that  $\hat{\mathcal{N}}(k^{-2}) \rightarrow 0$  as  $k \rightarrow +\infty$ , hence  $\hat{\mathcal{N}}, \hat{\mathcal{M}}$  are not r.c., by Lemma 5.5(a1). Moreover,  $\hat{\mathcal{M}}(s) = \prod_{n=2}^\infty |1 - 2/(1 + s/n^2)| \geq \prod_{n=2}^\infty |1 - 2n^{-2}| < \infty$  when  $\text{Re } s > 1$ , because  $\sum_n 2n^{-2} < \infty$ . Thus,  $\|\hat{\mathcal{M}}(s)^{-1}\|$  is bounded on  $\mathbb{C}_1^+$ .  $\square$

We note that a right factorization  $\mathcal{N}\mathcal{M}^{-1}$  is a q.r.c.f. iff any  $\mathcal{GTIC}_\infty(U)$  common right factor of  $\mathcal{N}$  and  $\mathcal{M}$  is a unit (i.e., iff  $[\frac{\mathcal{N}}{\mathcal{M}}] = [\frac{\mathcal{N}_0}{\mathcal{M}_0}]\mathcal{E}$ ,  $\mathcal{N}_0, \mathcal{M}_0, \mathcal{E} \in \text{TIC}$ ,  $\mathcal{E} \in \mathcal{GTIC}_\infty(U) \Rightarrow \mathcal{E} \in \mathcal{GTIC}$ ). (Proof: apply Lemma 5.5(d) to any q.r.c.f. of  $\mathcal{N}\mathcal{M}^{-1}$ .)

If the requirement  $\mathcal{GTIC}_\infty$  were dropped, the above condition would be the definition of “weakly coprime” in [S89]. (Thus, any “w.r.c.f.” is a q.r.c.f.; let  $\mathcal{E}$  be the right shift on  $U := \ell^2(\mathbb{N})$  to observe that no maps are “w.r.c.” if  $\dim U = \infty$ .)

If the system has more poles than its transfer function, no  $\mathcal{U}_{\text{exp}}$ -stabilizing state feedback can be right coprime, as illustrated in Example 6.4 below. However, if the system is estimatable, then this is not the case (and  $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$ ), by Corollary 5.7.

The “ $\mathcal{U}_{\text{exp}}$ -variant” of Theorem 5.9 is contained in Corollary 5.2 except that (iii) must be dropped, by Example 6.4. Similarly, Corollary 5.13 has an  $\mathcal{U}_{\text{exp}}$ -variant:

**Corollary 5.15** ( $\mathcal{U}_{\text{exp}} : \mathcal{N}\mathcal{M}^{-1}$ ) *The following are equivalent for any  $\mathcal{D} \in \text{TIC}_\infty(U, Y)$ :*

- (i)  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$ , where  $\mathcal{N}, \mathcal{M} \in \text{TIC}_{\text{exp}}$ ,  $\mathcal{M} \in \mathcal{GTIC}_\infty(U)$ .
- (iii)  $\mathcal{D}$  has an optimizable realization (i.e., one with  $\mathcal{U}_{\text{exp}}(x_0) \neq \emptyset \forall x_0 \in H$ ).
- (iv) For some  $\omega \in \mathbb{R}$ ,  $\delta < 0$ , and any  $v \in L_\omega^2(\mathbb{R}_-; U)$ , there exists  $u \in L_\delta^2(\mathbb{R}_+; U)$  s.t.  $\pi_+ \mathcal{D}(v + u) \in L_\delta^2$ .

Here  $\text{TIC}_{\text{exp}} := \cup_{\omega < 0} \text{TIC}_\omega$ . Note that by shifting (Remark 6.1.9 of [M02]) we obtain some kind of “exponential” version of any of the  $\mathcal{U}_{\text{out}}$  results of (e.g.) this section.

**Proof:** By Lemma 6.6.29 of [M02] (which was explained in the proof of Example 5.11), (i) implies (iii). The converse follows from Corollary 5.2 (set  $\mathcal{M} := (I - \mathcal{F})^{-1}$ ). We get “(iv)  $\Leftrightarrow$  (i)” from Corollary 5.13 applied to  $\hat{\mathcal{D}}(\cdot + \delta)$  (or  $e^{-\delta \cdot} \mathcal{D}e^{\delta \cdot}$ ) (for each  $\delta < 0$ ).  $\square$

Next we present the  $\mathcal{U}_{\text{out}}$ -variant of Corollary 5.3. The “optimal” output-stabilizing feedback for  $\Sigma$  also output-stabilizes any extension of  $\Sigma$ :

**Corollary 5.16** ( $\mathcal{U}_{\text{out}}$  through  $B_1$ ) *Assume the output-FCC and choose  $[\mathcal{K} \mid \mathcal{F}]$  as in Theorem 5.9(iii). If  $\tilde{\Sigma} := [\frac{\mathcal{A} \mid \mathcal{B} \mid \mathcal{H}}{\mathcal{C} \mid \mathcal{D} \mid \mathcal{G}}]$  is a WPLS (say, on  $(U \times W, H, Y)$ ), for some  $\mathcal{H}, \mathcal{G}$ , then there is  $\mathcal{E} \in \text{TIC}_\infty(W, U)$  s.t.  $[\tilde{\mathcal{K}} \mid \tilde{\mathcal{F}}] := [\mathcal{K} \mid \mathcal{F} \mathcal{E}]$  is a SOS-stabilizing state-feedback pair for  $\tilde{\Sigma}$ .*

Moreover,  $\tilde{\mathcal{N}}, \tilde{\mathcal{M}}$  are q.r.c., and so are  $\mathcal{D}_\circ = \tilde{\mathcal{D}}_\circ[I_0]$  and  $\mathcal{M} = \tilde{\mathcal{M}}_{11}$ .

(The proof is given on p. 62. Note that  $\tilde{\mathcal{K}}_\odot = \begin{bmatrix} \mathcal{K}_\odot \\ 0 \end{bmatrix}$ ,  $\tilde{\mathcal{M}} = \begin{bmatrix} \mathcal{M} & \mathcal{M}^\mathcal{E} \end{bmatrix}$ ,  $\tilde{\mathcal{F}}_\odot = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$ .) If  $[\mathcal{K} \mid \mathcal{F}]$  is given by some state-feedback operator  $K : \text{Dom}(A) \rightarrow U$ , then the above corollary surprisingly tells us that not only is  $K$  “compatible” with  $H$  but it also makes  $\mathcal{G}_\odot$  stable.

The above corollary leads to the  $\mathcal{U}_{\text{out}}$ -variant of Theorem 5.7, showing that the output-FCC for  $\Sigma$  and  $\Sigma^d$  is sufficient for the existence of a d.c.f. of  $\mathcal{D}$ :

**Theorem 5.17** ( $\mathcal{U}_{\text{out}} \& \mathcal{U}_{\text{out}}^{\Sigma^d} \Rightarrow \text{d.c.f.}$ ) *Assume that  $\Sigma$  and  $\Sigma^d$  satisfy the output-FCC. Let  $[\mathcal{K} \mid \mathcal{F}]$  and  $[\mathcal{K}^d \mid \mathcal{G}^d]$  be the corresponding optimal state-feedback pairs. Then they are jointly externally stabilizing and define a doubly coprime factorization of  $\mathcal{D}$ , namely (35).*

*Moreover, then any SOS- (resp. I/O-)stabilizing state-feedback pair for  $\Sigma$  is externally (resp. I/O-)r.c.-stabilizing.*

*Finally, the equivalence of Corollary 5.7 also holds after replacements “external”  $\mapsto$  “exponential” and “state-FCC”  $\mapsto$  “output-FCC” (again “with internal loop” is extraneous if  $\dim U, \dim Y < \infty$ ).*

(The proof is given on p. 62. A system is *externally stable* if its components are stable except possibly the semigroup. Thus, the two pairs are *jointly externally stabilizing* if there exists  $\mathcal{E} \in \text{TIC}_\infty(Y, U)$  s.t. (34) is a WPLS, and  $(\Sigma_{\text{Joint}})_L$  and  $(\Sigma_{\text{Joint}})_L^d$  are externally stable (i.e., their components, except possibly  $\mathcal{A}_L, \mathcal{A}_L^d$ , are stable).)

In particular, the  $\text{TIC}(U \times Y)$  maps in (35) are stable and the inverses of each other.

In the theorem, one can replace  $[\mathcal{K} \mid \mathcal{F}]$  by any other SOS-stabilizing pair (or by any other I/O-stabilizing pair if  $(\Sigma_{\text{Joint}})_L$  is required to be merely I/O-stable, i.e., to have its I/O map in  $\text{TIC}$ ), as one observes from the proof.

We finish this section by giving “generalizations” of Corollaries 5.2 and 5.3 and further observations. The following result shows that the optimization over a typical domain of optimization can be completely reduced to the optimization of a SOS-stable system over  $L^2(\mathbb{R}_+; U)$  (a similar claim on partial control is given in Proposition 5.19):

**Proposition 5.18** ( $\mathcal{U}_* : \text{FCC} \Leftrightarrow \text{stabilizable}$ ) *Assume that  $\vartheta = 0$  and  $Z^s = L^2(\mathbb{R}_+; \tilde{Y})$ . Then the FCC is satisfied iff there is a state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$  s.t.  $\mathcal{K}_\odot x_0 \in \mathcal{U}_*(x_0) \forall x_0 \in H$ . Assume the FCC and choose  $[\mathcal{K} \mid \mathcal{F}]$  as in the proof.*

(a) *Then  $\mathcal{K}_\odot x_0 + \mathcal{M} u_\odot \in \mathcal{U}_*(x_0) \Leftrightarrow u_\odot \in L^2(\mathbb{R}_+; U) \forall x_0 \in H \forall u_\odot$ ; in particular,  $[\mathcal{K} \mid \mathcal{F}]$  is SOS-stabilizing,  $\mathcal{X} \in \mathcal{GB}(\mathcal{U}_*(0), L^2(\mathbb{R}_+; U))$ .*

(b) *If  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  (resp.  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$ ), then  $\mathcal{N}, \mathcal{M}$  are q.r.c. (resp.  $\Sigma_\odot$  is exponentially stable), and also the pair of Theorem 5.9 (resp. Corollary 5.2) satisfies (a)–(d).*

(c) *The system  $\Sigma_2 := \begin{bmatrix} \mathcal{A}_\odot & \mathcal{B}_\odot \\ \mathcal{C}_\odot & \mathcal{D}_\odot \end{bmatrix}$  is [positively]  $J$ -coercive over  $\mathcal{U}_{\text{out}}^{\Sigma_2} = L^2(\mathbb{R}_+; U)$  iff  $\Sigma$  is [positively]  $J$ -coercive over  $\mathcal{U}_*$ .*

(d) *A control  $u_\odot$  is  $J$ -critical for  $x_0, \Sigma_2$  and  $J$  over  $\mathcal{U}_{\text{out}}^{\Sigma_2}$  iff  $u := \mathcal{K}_\odot x_0 + \mathcal{M} u_\odot$  is  $J$ -critical for  $x_0, \Sigma$  and  $J$  over  $\mathcal{U}_*$ . Moreover,  $\mathcal{J}(x_0, u) = \mathcal{J}_\odot(x_0, u_\odot) := \langle y_\odot, J y_\odot \rangle$ ,  $y_\odot := \mathcal{C}_\odot x_0 + \mathcal{D}_\odot u_\odot \forall x_0, u_\odot$ , hence  $\mathcal{P} = \mathcal{P}_2$  (if either, hence both exist).*

This shows that the inputs  $u \in \mathcal{U}_*(x_0)$  correspond 1-1 to the stable inputs to the stabilized system. Sometimes this allows one to reduce the problem to Theorem 5.26 or other results for the stable case. This is particularly useful when one can show that the smoothness is preserved in the uniformly positive case (cf. Section 8), even if the original problem were indefinite. However, often one prefers to use the original data instead of  $\Sigma_2$ .

**Proof of Proposition 5.18:** 1° “Iff”, SOS,  $\mathcal{X}$ , q.r.c.: Define  $\tilde{\Sigma}$  by setting  $\tilde{\mathcal{A}} := \mathcal{A}$ ,  $\tilde{\mathcal{B}} := \mathcal{B}$ ,  $\tilde{\mathcal{C}} := \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix}$ ,  $\tilde{\mathcal{D}} := \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix}$ . Since, obviously,  $\mathcal{U}_{\text{out}}^{\tilde{\Sigma}} = \mathcal{U}_*$ , we obtain the equivalence from Theorem 5.9 (whose proof shows that  $\tilde{\mathcal{P}} \geq 0$ ,  $\tilde{\mathcal{S}} = I$ ), by which  $\tilde{\Sigma}_\odot$  is also SOS-stable, hence so is  $\Sigma_\odot$  (being contained in  $\tilde{\Sigma}_\odot$ ).

By Lemma 12.2(c),  $\mathcal{X} \in \mathcal{GB}(\mathcal{U}_*(0), L^2(\mathbb{R}_+; U))$  and  $\tilde{\mathcal{D}}_\odot$  and  $\mathcal{M}$  are q.r.c., hence so are  $[\tilde{\mathcal{A}}_\odot]$  and  $\mathcal{M}$ , because  $\tilde{\mathcal{D}}_\odot = \begin{bmatrix} \mathcal{A}_\odot \\ \mathcal{M} \end{bmatrix}$  (hence so are  $\mathcal{N}$  and  $\mathcal{M}$  if  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  so that  $\mathcal{R}_\odot = \mathcal{N}$ ).

2° “ $\Leftarrow$ ”: Given  $u_\odot \in L^2_{\text{loc}}(\mathbb{R}_+; U)$ , define  $u$  as in Lemma 3.8. If  $u_\odot \in L^2$ , then  $u = \tilde{y}_\odot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in L^2$ . If  $u \in \mathcal{U}_*(x_0)$ , then  $u = \mathcal{K}_\odot x_0 + \mathcal{M} u_\odot$ , hence  $\mathcal{M} u_\odot \in \mathcal{U}_*(0)$ , by Lemma 4.2, hence  $u_\odot = \mathcal{X} \mathcal{M} u_\odot \in L^2$ .



(b) The “q.r.c.” claim was proved in 1°. If  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$ , then  $\tilde{\Sigma}_{\circ} \begin{bmatrix} I \\ 0 \end{bmatrix}$  is exponentially stable, by Theorem 4.7, hence so is  $\tilde{\Sigma}_{\circ}$ , by Lemma 2.2. For the pair of the theorem or the corollary, the proofs are similar than to that above (and mostly given in Theorem 8.4.5 of [M02]).

(c)&(d) These follow easily from Lemma 3.8 and the claims on  $\mathcal{X}$  and  $u_{\circ}$  in (a).  $\square$

If  $\Sigma$  is formally stabilizable through the first input, then it is (state-feedback) stabilizable through the first input:

**Proposition 5.19 ( $\mathcal{U}_*$ : stabilizable through  $B_1$ )** *Assume that  $\vartheta = 0$  and  $Z^s = L^2(\mathbb{R}_+; \tilde{Y})$  and that  $[\mathcal{K} \mid \mathcal{F}]$  is as in Proposition 5.18. If  $\begin{bmatrix} \mathcal{A} + \mathcal{B}\mathcal{K} \\ \mathcal{G} \end{bmatrix}$  is a WPLS (for some  $\mathcal{H}, \mathcal{G}, \mathcal{T}$ ), then  $[\mathcal{K} \mid \mathcal{F}]$  can be extended as in Corollary 5.16,*

Note that  $\tilde{\mathcal{K}}_{\circ} x_0 + \tilde{\mathcal{M}} \begin{bmatrix} u_1 \\ w \end{bmatrix} = \begin{bmatrix} u_1 \\ w \end{bmatrix}$ , and  $w = 0 \Rightarrow (u_1 \in \mathcal{U}_*(x_0) \Leftrightarrow u_{\circ} \in L^2(\mathbb{R}_+; U))$ . Obviously,  $\begin{bmatrix} u_1 \\ w \end{bmatrix} \in \mathcal{U}_{\text{out}}^{\tilde{\Sigma}}(x_0) \Leftrightarrow \begin{bmatrix} u_1 \\ w \end{bmatrix} \in L^2$  (this is useful for  $H^{\infty}$  problems, whether over  $\mathcal{U}_{\text{out}}$ ,  $\mathcal{U}_{\text{exp}}$  or something else). To be brief, we shall postpone the obvious further equivalents of (a)–(d) of Proposition 5.19 to an  $H^{\infty}$  article.

**Proof of Proposition 5.19:** Apply Corollary 5.16 to the  $\tilde{\Sigma}$  of the proof of Proposition 5.18. Then  $\tilde{\mathcal{N}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \tilde{\mathcal{M}}$  and  $\tilde{\mathcal{M}}$  are q.r.c., hence so are  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \tilde{\mathcal{M}}$  and  $\tilde{\mathcal{M}}$  (which are the “ $\tilde{\mathcal{N}}$ ,  $\tilde{\mathcal{M}}$ ” of Corollary 5.16 applied to the system in Proposition 5.19).  $\square$

It is known that a matrix-valued transfer function has a stabilizing dynamic feedback controller iff it has a d.c.f. Using Corollary 5.13, we can extend “only if” to operator-valued proper transfer functions:

**Lemma 5.20** *If  $\mathcal{D}$  has a stabilizing dynamic feedback controller (without an internal loop), i.e.,  $\begin{bmatrix} I & -\mathcal{T} \\ -\mathcal{D} & I \end{bmatrix}^{-1} \in \text{TIC}$  for some  $\mathcal{T} \in \text{TIC}_{\infty}(Y, U)$ , then  $\mathcal{D}$  has a d.c.f.*

It follows that all stabilizing dynamic feedback controllers for  $\mathcal{D}$  are given by the standard Youla parametrization formula (p. 290 of [M02]). Conversely, if  $\mathcal{D}$  has a d.c.f. and  $\dim U, \dim Y < \infty$ , then  $\mathcal{D}$  has a *stable* stabilizing dynamic feedback controller, by Corollary 6.6 of [Q03]. The case for general  $U, Y$  is still open (whether “stable” removed or not). We have named the above result a lemma, since it is a special case of Theorem 5.21 (and needed for its proof).

**Proof:** We have the right factorizations  $\mathcal{D} = \mathcal{N}_0 \mathcal{M}_0^{-1}$ ,  $\mathcal{T} = \mathcal{Y}_0 \mathcal{X}_0^{-1}$ , where  $\mathcal{M}_0 = (I - \mathcal{T} \mathcal{D})^{-1}$ ,  $\mathcal{X}_0 = (I - \mathcal{D} \mathcal{T})^{-1}$ , by (7.5) of [M02]. Therefore, there are q.r.c.f.’s  $\mathcal{D} = \mathcal{N} \mathcal{M}^{-1}$ ,  $\mathcal{T} = \mathcal{Y} \mathcal{X}^{-1}$ , by Corollary 5.13. By Lemma 7.1.5(b) of [M02],  $\begin{bmatrix} \mathcal{N} & \mathcal{Y} \\ \mathcal{M} & \mathcal{X} \end{bmatrix} \in \mathcal{GTIC}$ , i.e.,  $\mathcal{D}$  and  $\mathcal{T}$  have a (joint) d.c.f. (We used above the fact that the proof of Lemma 7.1.5 (and 6.6.6) obviously applies to q.r.c.f.’s in place of r.c.f.’s.)  $\square$

This leads to the following equivalence for any  $\mathcal{D} \in \text{TIC}_{\infty}(U, Y)$  (the terminology will be explained below):

**Theorem 5.21 (D.c.f.  $\Leftrightarrow$  ...)** *The following are equivalent for any  $\mathcal{D} \in \text{TIC}_{\infty}(U, Y)$ :*

- (i)  $\mathcal{D}$  has a d.c.f.
- (ii)  $\mathcal{D}$  has a r.c.f.
- (iii)  $\mathcal{D}$  has a l.c.f.
- (iv)  $\mathcal{D}$  has a realization  $\Sigma$  s.t. the output-FCC holds for  $\Sigma$  and  $\Sigma^d$ .
- (v)  $\mathcal{D}$  has a stabilizable and detectable realization.
- (vi)  $\mathcal{D}$  has a jointly stabilizable and detectable realization.
- (vii)  $\mathcal{D}$  has a stabilizing controller with internal loop.
- (viii)  $\mathcal{D}$  has a stabilizing canonical controller.
- (ix) Some realization of  $\mathcal{D}$  has a stabilizing controller with internal loop.
- (x)  $\begin{bmatrix} \mathcal{D} & 0 \\ 0 & I \end{bmatrix}$  has a d.c.f. (or r.c.f. or l.c.f.).

*If  $\dim U, \dim Y < \infty$ , then we have three more equivalent conditions:*

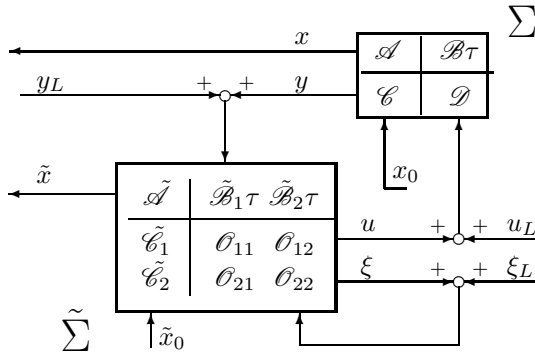


Figure 4: DF-controller  $\tilde{\Sigma}$  with internal loop for  $\Sigma \in \text{WPLS}(U, H, Y)$

- (xi)  $\mathcal{D}$  has a stabilizing controller.
- (xii) Some realization of  $\mathcal{D}$  has a stabilizing controller.
- (xiii)  $\hat{\mathcal{D}} = FG^{-1}$  with  $F, G \in H^\infty$ ,  $F^*F + G^*G \geq \epsilon I$  on  $\mathbb{C}^+$  for some  $\epsilon > 0$ ,  $\det G \neq 0$ .

Given a d.c.f., all stabilizing controllers with internal loop for  $\mathcal{D}$  are obtained from the standard Youla parameterization  $(\mathcal{Y} + \mathcal{M}\mathcal{E})(\mathcal{X} + \mathcal{N}\mathcal{E})^{-1}$ , where  $\mathcal{E} \in \text{TIC}(Y, U)$  is arbitrary (the controller is proper iff  $\mathcal{X} + \mathcal{N}\mathcal{E} \in \mathcal{GTIC}_\infty(U)$ ).

In particular, any stabilizing controller with internal loop (for any well-posed map) is equivalent to a canonical controller.

(Any dimension of  $I$  will do in (x). In general (xi) and (xii) are sufficient and the Corona condition (xiii) necessary but not sufficient.)

We say that  $\mathcal{O}$  is a *stabilizing controller with internal loop* for  $\mathcal{D}$  if  $\mathcal{O} \in \text{TIC}_\infty(Y \times \Xi, U \times \Xi)$  for some Hilbert space  $\Xi$  and  $(I - \mathcal{D}^o)^{-1} \in \text{TIC}$ , where  $\mathcal{D}^o = \begin{bmatrix} 0 & \mathcal{O}_{11} & \mathcal{O}_{12} \\ 0 & \mathcal{O}_{21} & \mathcal{O}_{22} \end{bmatrix}$ . Note from Figure 4 that  $\mathcal{D}_I^o : \begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \\ \xi \end{bmatrix}$ , where  $\mathcal{D}_I^o := (I - \mathcal{D}^o)^{-1} - I$ ; cf. Definition 3.1 with  $L = I$ . Thus,  $\mathcal{O}$  is stabilizing iff the maps  $\begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \\ \xi \end{bmatrix}$  are well-posed and stable.

Condition (ix) is formally stronger: it means (the existence of  $\Sigma$  and  $\tilde{\Sigma}$  for this fixed  $\mathcal{D}$  such) that all 25 maps from initial states and external inputs to states and outputs in Figure 4 are stable (i.e., that  $\Sigma_I^o$  is stable, where  $\Sigma^o$  is given by (7.21) of [M02]). See Section 7.2 of [M02] for further details.

If  $\mathcal{Y} \in \text{TIC}(Y, U)$  and  $\mathcal{X} \in \text{TIC}(U)$  are r.c., then  $\mathcal{O} := \begin{bmatrix} 0 & \mathcal{Y} \\ I & I - \mathcal{X} \end{bmatrix}$  is called a *canonical controller* (see [CWW01] or [M02]); in [M02], the term *controller with a coprime internal loop* was used. Sometimes we denote it by  $\mathcal{Y}\mathcal{X}^{-1}$ , as in the Youla parameterization above. (It is equivalent to  $\begin{bmatrix} 0 & I \\ \mathcal{Y} & I - \mathcal{X} \end{bmatrix}$  for certain l.c.  $\tilde{\mathcal{Y}}$  and  $\tilde{\mathcal{X}}$ .)

A (dynamic feedback) controller  $\mathcal{O}$  (resp.  $\tilde{\Sigma}$ ) with internal loop is *proper* or *well-posed* (i.e., “with internal loop” can be dropped) iff  $I - \mathcal{O}_{22} \in \mathcal{GTIC}_\infty$ . In that case we can redefine  $\mathcal{O}$  (resp.  $\tilde{\Sigma}$ ) so as to have  $\mathcal{O}_{12}, \mathcal{O}_{21}, \mathcal{O}_{22} = 0$  (resp.  $\mathcal{O}_{12}, \mathcal{O}_{21}, \mathcal{O}_{22}, \tilde{\mathcal{B}}_2, \tilde{\mathcal{C}}_2 = 0$ ), as in the classical definition of a controller.

The Youla parameterization covers all stabilizing controllers with internal loop in the sense that any other controller with internal loop defines the same closed-loop map  $\begin{bmatrix} u_L \\ y_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}$  as exactly one of these (modulo  $\begin{bmatrix} \mathcal{Y}' \\ \mathcal{X}' \end{bmatrix} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix} \mathcal{E}$  for some  $\mathcal{E} \in \mathcal{GTIC}$ ), although the maps from  $\xi_L$  and to  $\xi$  (internal loops) may differ. In particular, this parameterization contains all well-posed stabilizing controllers.

Any map having a right factorization has a realization that satisfies the output-FCC, by Lemma 6.6.29 of [M02]. Thus, the map  $\mathcal{N}\mathcal{M}^{-1} = \mathcal{M}^{-1}\mathcal{N} \in \text{TIC}_1(\mathbb{C})$  of Example 5.14 has realizations that satisfy the output-FCC and ones whose dual satisfies the output-FCC. However, none of those realizations satisfies both, by (iv) and (ii) above.

**Proof of Theorem 5.21:** 1° We have  $(vi) \Leftrightarrow (vi') \Leftrightarrow (v) \Leftrightarrow (iv) \Leftrightarrow (i)$ : By Theorem 5.17, we have  $(vi') \Leftrightarrow (iv) \Rightarrow (i)$ , where we have added “externally” to (vi) to define (vi'). The equivalence of (i) and (vi) was established in Theorem 4.4 of [S98a]. The implications “(vi')  $\Leftarrow$  (vi)  $\Rightarrow$  (v)  $\Rightarrow$  (iv) are obvious.

2°  $(vii) \Rightarrow (x)$ : Assume (vii). By Lemma 7.2.6 of [M02], some  $\mathcal{O} \in \text{TIC}_\infty$  (I/O-)stabilizes  $\underline{\mathcal{D}} := \begin{bmatrix} \mathcal{D} & 0 \\ 0 & I \end{bmatrix}$ . By Lemma 5.20, (x) follows.

3°  $(x) \Rightarrow (iv)$ : Apply “(i)  $\Leftrightarrow$  (v)” to obtain a stabilizable and detectable realization of  $\underline{\mathcal{D}}$ , and then remove the last row and column to satisfy (iv) for  $\mathcal{D}$ .

4°  $(i) \Rightarrow (ii) \Rightarrow (viii) \Rightarrow (vii)$ : Implications “(i) $\Rightarrow$ (ii)” and “(viii) $\Rightarrow$ (vii)” are trivial, and “(ii) $\Rightarrow$ (viii)” is from Corollary 7.2.13(b) of [M02].

5° From the above we see that (i), (ii), (iv), (v), (vi), (vii), (viii) and (x) are equivalent. By duality, we get “(i) $\Leftrightarrow$ (iii)”.

6°  $(vii) \Leftrightarrow (ix)$ : Implication “(ix) $\Rightarrow$ (vii)” is trivial. Conversely, if (vii) holds, then  $\mathcal{D}$  and  $\mathcal{O}$  (see 2°) have d.c.f.’s, hence (vi) holds to both of them. Therefore, (ix) follows from Theorem 7.2.3(b)(3.) of [M02].

7°  $(ii) \Rightarrow (xiii) \Rightarrow (vii)$ : We have (ii) $\Rightarrow$ (xi) in general, by Lemma 5.5(a1). Conversely, if  $\dim U < \infty$  and (xi) holds (it tacitly requires that  $G \in H^\infty(\mathbb{C}^+; \mathcal{B}(U))$ ), then  $[\frac{G}{F}]$  can be extended to  $R := [\frac{G}{F} \frac{\mathcal{D}}{\mathcal{F}}] \in \mathcal{GH}^\infty(\mathbb{C}^+; \mathcal{B}(U \times Y))$ , hence  $\hat{\mathcal{D}} \hat{\mathcal{X}}^{-1}$  satisfies (vii) (since  $(I - \mathcal{D}_I^o)^{-1} \in \text{TIC} \Leftrightarrow (\hat{\mathcal{X}} - FG^{-1}\hat{\mathcal{D}}) \in \text{TIC} \Leftrightarrow R \in \mathcal{GH}^\infty$ ).

8°  $(xi) \mathcal{E} (xii)$ : Obviously, (xi) or (xii) implies (ix) or (vii). Assume then (i). As noted below Lemma 5.20, (vii) holds even without “with internal loop”; so does (ix) too, by 2° (slightly modified).

9° *All stabilizing controllers*: This follows from Theorem 7.2.14(ii) of [M02].

10° *Canonical controllers*: The last claim follows now from Corollary 7.2.13(a1) of [M02].  $\square$

The “ $\mathcal{U}_{\text{exp}}$ -variant” of Theorem 5.9 is contained in Corollary 5.2 except that (iii) must be dropped, by Example 6.4. Similarly, Theorem 5.21 has an  $\mathcal{U}_{\text{exp}}$ -variant:

**Corollary 5.22 (Exponential d.c.f.)** *The following are equivalent for any  $\mathcal{D} \in \text{TIC}_\infty(U, Y)$ :*

- (i)  $\mathcal{D}$  has an exponential d.c.f.
- (ii)  $\mathcal{D}$  has an exponentially stabilizing controller with internal loop.
- (iii)  $\mathcal{D}$  satisfies the exponential version of any (hence all) of (i)–(x) of Theorem 5.21.
- (iv)  $\mathcal{D}$  has a realization that satisfies any (hence all) of (i)–(v) of Corollary 5.7.  $\square$

An *exponential d.c.f.* is defined by Definition 5.4(b2) with  $\mathcal{GTIC}_{\text{exp}}$  in place of  $\mathcal{GTIC}$  (here  $\text{TIC}_{\text{exp}} := \cup_{\omega < 0} \text{TIC}_\omega$ ). By (iii)(ii), it is equivalent to an exponential r.c.f. (or l.c.f.).

Thus, (i) holds iff there exists  $\omega < 0$  s.t.  $e^{-\omega \cdot} \mathcal{D} e^{\omega \cdot}$  (which is the I/O map corresponding to  $\hat{\mathcal{D}}(\cdot + \omega)$ ) has a d.c.f. That is the exponential version of Theorem 5.21(i) (see Remark 6.1.9 of [M02] for details on shifting), hence equal to (iii)(i). Similarly, (ii) equals (iii)(vii) and (iii)(vi) equals (iv)(i); this proves Corollary 5.22.

For matrix-valued transfer functions, one typically allows for any controllers  $\hat{\mathcal{T}} := fg^{-1}$  or  $g^{-1}f$  (“ $H^\infty/H^\infty$ ” fraction controllers, possibly improper), where  $f, g \in H^\infty$  and  $\det g \neq 0$ . We recall from Remark 7.2.8 of [M02] that such controllers (and more) are covered by controllers with internal loop:

**Remark 5.23 ( $H^\infty/H^\infty$  controllers)** *Let  $\Xi$  be a Hilbert space, and let  $f \in H^\infty(Y \times \Xi, U \times \Xi)$  be s.t.  $f_{22}$  is invertible at some  $s_0 \in \mathbb{C}^+$ . The map  $\hat{\mathcal{T}} := f_{11} + f_{12}f_{22}^{-1}f_{21}$  “stabilizes  $\hat{\mathcal{D}}$ ” (i.e.,  $[\frac{I}{-\hat{\mathcal{D}}} \frac{-\hat{\mathcal{T}}}{I}]$  equals the inverse of a  $\text{TIC}(U \times Y)$  map near  $s_0$ ) iff the map  $\mathcal{O} \in \text{TIC}_\infty$ , defined by  $\hat{\mathcal{O}} := [\frac{f_{11}}{f_{21}} \frac{f_{12}}{I - f_{22}}]$ , is a stabilizing controller with internal loop for  $\mathcal{D}$ . If this is the case, then  $\mathcal{D}$  has a d.c.f. and  $\mathcal{O}$  is “equivalent to  $\mathcal{T}$ ”.*

*Naturally, this remark also holds with “DPF-stab” in place of “stab” (see the end of Corollary 5.24) if we remove the “i.e.”-comment in parenthesis.*

(This follows from the computations of the proof of Lemma 7.2.7 of [M02] as in Remark 7.2.8, except that the d.c.f. (which is “joint with  $\mathcal{T}$ ” due to the last claim of Theorem 5.21) is from Theorem 5.21.)

E.g., the finite-dimensional unstable SISO plant  $\hat{\mathcal{D}}(s) = 1 + 1/s$  is (exponentially) stabilized by the controller  $\mathcal{O} = [\frac{0}{-1} \frac{1}{1}]$  with internal loop (since  $I - \mathcal{O}_{22} = 0$  is nowhere invertible, this is not equivalent to any proper nor to any “ $H^\infty/H^\infty$ ” controller, by Remark 7.2.8 of [M02]). (As noted in p. 7 of [WC97], this example is physically meaningful.)

*Dynamic partial feedback (DPF)* of  $\mathcal{D} \in \text{TIC}_\infty(U \times W, Z \times Y)$ , where also  $W$  and  $Z$  are Hilbert spaces, means that, in Figure 4, there is an additional first output (“ $z$ ”) and second input (“ $w$ ”) that are not connected to the controller. Thus,  $\mathcal{O}$  is a stabilizing *DPF-controller* for  $\mathcal{D}$  with internal loop iff  $\mathcal{O}_{\text{DF}} := [\frac{0}{0} \frac{\mathcal{O}_{11}^o}{\mathcal{O}_{21}^o} \frac{\mathcal{O}_{12}^o}{\mathcal{O}_{22}^o}]$  is a stabilizing (DF-)controller for  $\mathcal{D}$  with internal loop (see Section 7.3 of [M02] for details).

**Corollary 5.24 (Partial feedback)** *The following are equivalent for  $\mathcal{D} \in \text{TIC}_\infty(U \times W, Z \times Y)$ :*

- (i)  $\mathcal{D}$  has a stabilizing DPF-controller with internal loop
- (ii)  $\mathcal{D}$  has a r.c.f. of the form  $\mathcal{D} = \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ \mathcal{N}_{21} & \mathcal{N}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ 0 & I \end{bmatrix}^{-1}$  s.t.  $\mathcal{N}_{21}$  and  $\mathcal{M}_{11}$  are r.c.
- (iii)  $\mathcal{D}$  has a l.c.f. of the form  $\mathcal{D} = \begin{bmatrix} I & \tilde{\mathcal{M}}_{12} \\ 0 & \tilde{\mathcal{M}}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_{11} & \tilde{\mathcal{N}}_{12} \\ \tilde{\mathcal{N}}_{21} & \tilde{\mathcal{N}}_{22} \end{bmatrix}$  s.t.  $\tilde{\mathcal{N}}_{21}$  and  $\tilde{\mathcal{M}}_{22}$  are l.c.

*If this is the case, then such controllers are exactly the stabilizing controllers with internal loop for  $\mathcal{D}_{21}$ ; thus, any of them is equivalent to a canonical controller given by the Youla parameterization (in particular, some of them are proper if  $\dim U, \dim Y < \infty$ ).*

Note that  $\mathcal{D}_{21} : (u + u_L) \mapsto y$  is the control-to-measurement part of  $\mathcal{D}$ , and that  $\mathcal{D}_{21} = \mathcal{N}_{21} \mathcal{M}_{11}^{-1}$  is a r.c.f. (under (ii)). By Lemma 7.3.8 of [M02], two controllers with internal loop are equivalent as DPF for  $\mathcal{D}$  iff they are equivalent as DF for  $\mathcal{D}_{21}$ .

In particular, the stabilizing DPF-controllers for  $\mathcal{D}$  are exactly the canonical controllers  $\tilde{\mathcal{X}}^{-1} \tilde{\mathcal{Y}}$  for  $\tilde{\mathcal{X}}, \tilde{\mathcal{Y}} \in \text{TIC}$  s.t.  $\tilde{\mathcal{X}} \mathcal{M}_{11} - \tilde{\mathcal{Y}} \mathcal{N}_{21} = I$ , equivalently,  $\mathcal{Y} \mathcal{X}^{-1}$  for which  $\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{Y} \\ \mathcal{N}_{21} & \mathcal{N}_{22} & 0 \\ \mathcal{N}_{21} & \mathcal{N}_{22} & \mathcal{X} \end{bmatrix} \in \mathcal{GTIC}(U \times W \times Y)$  for some (hence all) r.c.f.  $\mathcal{N} \mathcal{M}^{-1}$  of  $\mathcal{D}$  (Lemma 7.3.22 of [M02]).

The coprimeness condition cannot be weakened: the (exponential) r.c.f.  $\hat{\mathcal{N}} \hat{\mathcal{M}}^{-1} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s/(s+1) & 0 \\ 0 & 1 \end{bmatrix}^{-1}$  is of the form  $\begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}^{-1}$  (hence  $\mathcal{D}$  and  $\mathcal{D}_{21}$  both have a d.c.f. and are thus DF-stabilizable with internal loop), but yet  $\mathcal{D}$  is not DPF-stabilizable with internal loop. However, if  $\mathcal{D}$  is DPF-stabilizable, then any r.c.f. of the form  $\begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}^{-1}$  has  $\mathcal{N}_{21}, \mathcal{M}_{11}$  r.c., by Corollary 7.3.17 of [M02].

By Corollary 5.24, Hypothesis 7.3.15 of [M02] holds iff  $\mathcal{D}$  is DPF-stabilizable with internal loop. These results simplify significantly Section 7.3 of [M02] like Theorem 5.21 did for 7.1 and 7.2.

(Note: the proof of “(ii) $\Rightarrow$ (iii)” in Lemma 7.3.6(b2) of [M02] has been written down incompletely. Perhaps the shortest way to prove Lemma 7.3.6 is to choose pair  $[\mathcal{K} \mid \mathcal{F}]$  for  $\Sigma_{21}$  (use Theorem 5.17) and extending it as in Corollary 5.16, to obtain (iii) above, hence (i) above, so that 7.3.6(b2)(iii) follows from 7.3.11(b)(1.). Alternatively, work as in Corollary 2.2 of [G92] and use (i) $\Leftrightarrow$ (ii) of Theorem 5.21.)

**Proof of Corollary 5.24:** If  $\mathcal{O}$  DPF-stabilizes  $\mathcal{D}$  with IL, then it DF-stabilizes  $\mathcal{D}_{21}$  with IL, by Lemma 7.3.5 of [M02], hence then  $\mathcal{D}_{21}$  has a d.c.f., by Theorem 5.21. Thus, (i) is equivalent to (i) (hence to (ii) and (iii) too) of Proposition 7.3.14 of [M02], whose proof provides the equivalence. The remaining claims follow from Theorem 7.3.19 (and 7.3.20) of [M02] (except case  $\dim U, \dim Y < \infty$  from Theorem 5.21).  $\square$

By combining the results of this section with Chapter 7 (including Lemmas 7.3.5 and 7.3.6(b1)) of [M02], we get the following:

**Corollary 5.25** *A WPLS  $\Sigma$  on  $(U \times W, H, Z \times Y)$  is exponentially DPF-stabilizable with internal loop iff  $\Sigma_{21} := \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}_{21} \end{bmatrix}$  and its dual satisfy the state-FCC, in which case the exponentially DPF-stabilizing controllers with internal loop for  $\Sigma$  equal the (DF-)stabilizing controllers with internal loop for  $\Sigma_{21}$ .*

*A similar claim holds with “exponentially” removed and “output-FCC” in place of “state-FCC” (cf. the proof of Theorem 5.21), except that then also  $\Sigma$  and  $\Sigma^d$  must satisfy the output-FCC.*  $\square$

Theorem 5.1 solved positively  $J$ -coercive problems by reducing them to the stable (positive) spectral factorization result given below. Indefinite problems for stable I/O maps of MTIC (convolutions with measures) type can be solved through spectral factorization as well, as explained in [M02].

**Theorem 5.26 (SpF)** (a) *If  $\mathcal{D}^* J \mathcal{D} \gg 0$ , then  $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* \mathcal{X}$  for some  $\mathcal{X} \in \mathcal{GTIC}(U)$ .*

(b) *If  $\pi_+ \mathcal{D}^* J \mathcal{D} \pi_+ \in \mathcal{GB}(\text{L}^2(\mathbb{R}_+; U))$  and  $\mathcal{D} \in \text{MTIC}^{\text{L}^1}(U, Y)$  then  $\mathcal{D}^* J \mathcal{D} = \mathcal{X}^* S \mathcal{X}$ , where  $\mathcal{X} \in \text{GMTIC}^{\text{L}^1}(U, Y)$ ,  $S \in \mathcal{GB}(U)$ .*

The claim  $\pi_+ \mathcal{D}^* J \mathcal{D} \pi_+ \geq \epsilon I$  (on  $\text{L}^2(\mathbb{R}_+; U)$ ) is equivalent to  $\mathcal{D}^* J \mathcal{D} \geq \epsilon I$  (on  $\text{L}^2(\mathbb{R}; U)$ ) (by Lemma 13(i) of [S97]).

By  $\mathcal{D} \in \text{MTIC}^{\text{L}^1}(U, Y)$  we mean that  $\mathcal{D}u = Du + f * u \ \forall u \in \text{L}^2$  for some  $D \in \mathcal{B}(U, Y)$ ,  $f \in \text{L}^1(\mathbb{R}_+; \mathcal{B}(U, Y))$  (i.e.,  $\mathcal{D}$  consists of an  $\text{L}^1$  impulse response plus a feedthrough). Similar results hold when  $\mathcal{D}$  also has delays (see Theorems 5.2.7–5.2.8 of [M02]). Even in the positive case, it is often important to use the fact that the MTIC classes are closed under spectral factorization; see, e.g., Sections 5.2, 8.4 and 9.1 of [M02] and Section 8 for details.

**Proof of Theorem 5.26:** (Instead of our standing hypothesis (4.1), it would suffice that  $\mathcal{D} \in \text{TIC}(U, Y)$ ,  $J = J^* \in \mathcal{B}(Y)$ .) The results are contained in Theorems 5.2.7–5.2.8 of [M02]. Alternatively, claim (a) can also be found from Lemma 18(ii) of [S97] (or from Lemma 5.2.1(a) of [M02]), and claim (b) is essentially given in [GL73].  $\square$

In Corollary 7.5 we shall show roughly that, in most results of this section, one more equivalent condition is that the corresponding IRE or ARE has a nonnegative solution, and that the smallest such solution provides the desired feedback or factorization. See also Sections 8 and 6 for the smoothness (or regularity) of the factorizations and closed-loop systems.

**Notes for Section 5:** Theorem 5.26(a) is essentially from [RR85] and (b) from [GL73]. We presented them and similar spectral factorization results (some of which were new) in Chapter 5 of [M02]. Definition 5.4 and Lemma 5.5 are from [M02]; except for quasi-coprimeness, they are well known (cf. [S98a]). Example 5.6 is due to Olof Staffans and Example 5.14 due to Sergei Treil.

We extended (the classical finite-dimensional version of) Corollary 5.7 to WPLSs having a smoothing semigroup ( $\mathcal{A}B, \mathcal{A}^*C^* \in \text{L}_{\text{strong,loc}}^1$ ,  $\mathcal{D} \in \text{ULR}$ ) in Theorem 7.2.4 of [M02], showing that then the dynamic controller in (v) does not need an internal loop (and giving its constructive formula). For finite-dimensional  $U$  and  $Y$ , Lemma 5.20 is implied by Theorem 1 of [S89], as shown in Lemma 7.1.4 of [M02].

All the other results of this section are new (except that the claim below (35) is from Theorem 4.4 of [S98a], as mentioned there). In particular, even for the cost  $\|y\|_2^2 + \|u\|_2^2$  with a bounded output operator ( $C \in \mathcal{B}(H, Y)$ ), as in [FLT88] (with  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$ ) it has not been known that the optimal state-feedback is well-posed, not even that it is admissible for the open-loop system.

The only exception is that we presented Corollary 5.2, Theorem 5.9(i)&(ii) and a weak version of Theorem 5.1 already in [M03]; at that time our proof was based on resolvent REs (a generalization of reciprocal REs, see [M03b]).

An thorough treatment of optimizability (and estimatability) is given in [WR00], although the concept (under the name FCC) is very old. Much of Propositions 5.18 and 5.19 was presented in [M02] (e.g., in Theorem 8.4.5).

We defined the concept “q.r.c.” in [M02], because q.r.c.-SOS-stabilization is the weakest form of stabilization that allows one to reduce problems (over  $\mathcal{U}_{\text{out}}$ ) to the stable case. (It is implied by r.c.-stabilization which in turn is implied by joint stabilization and detection.) Later it came to us as a surprise that q.r.c.-SOS-stabilization can be applied to all WPLSs for which  $\mathcal{U}_{\text{out}}$ -optimization makes sense (i.e., that (i) implies (iii) in Theorem 5.9).

## 6 Algebraic Riccati Equations (AREs)

Traditionally, one finds the optimal state-feedback by solving an ARE, such as (4) or (9). In this section we shall generalize this to weakly regular WPLSs (Definition 2.6). Since the equation becomes rather complicated in the general case, we shall show how it can be simplified in some special cases, the simplest of which is the (essentially known) case where  $B$  is bounded:

**Theorem 6.1 (Unique minimum  $\Leftrightarrow$  ARE)** *Assume that  $D^*JD \gg 0$  and  $B \in \mathcal{B}(U, H)$ . Then the following are equivalent:*

- (i) *There is a unique minimizing control over  $\mathcal{U}_{\text{exp}}(x_0)$  for each initial state  $x_0 \in H$ .*

(ii) *The (algebraic) Riccati equation (ARE)*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \\ S = D^*JD, \\ K = -S^{-1}(B^*\mathcal{P} + D^*JC), \end{cases} \quad (36)$$

has a solution  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$  that is exponentially stabilizing.

(iii) *The state-FCC (3) holds, and there is  $\epsilon > 0$  s.t. for all  $x_0 \in H$ ,  $u_0 \in U$ ,  $r \in \mathbb{R}$  we have*

$$(ir - A)x_0 = Bu_0 \Rightarrow \langle Cx_0 + Du, J(Cx_0 + Du) \rangle_Y \geq \epsilon \|x_0\|_H^2 \quad (37)$$

Assume that (ii) has a solution. Then this solution is unique, and the minimizing control is given by the state feedback  $u(t) = K_w x(t)$ . The minimal cost equals  $\langle x_0, \mathcal{P}x_0 \rangle_H$ .

□

(This is a special case of Theorem 1.2.6 of [M02]. See Theorem 11.2 for equivalent conditions to (37), one of which is that  $\mathcal{S}_{PT} \gg 0$ .)

By setting  $C := \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $D := \begin{bmatrix} 0 \\ I \end{bmatrix}$ ,  $J = I$ , we can make the cost equal to (2) and thus obtain Theorem 1.1 as a special case. The reader is invited to carry out the same simplification to most AREs presented in the sequel.

Naturally, without regularity the limit  $D := \hat{\mathcal{D}}(+\infty)$  does not exist and hence the ARE (36) becomes meaningless. Fortunately, all physically relevant WPLSs seem to be regular. When, e.g.,  $\mathcal{S}_{PT} \gg 0$  and that the FCC holds (and  $\vartheta = 0$ ), then there exists a  $J$ -optimal state-feedback pair  $[\mathcal{K} | \mathcal{F}]$ , by Theorem 5.1. In a generalization of Theorem 6.1, this pair should be given by a state-feedback operator, i.e.,  $\mathcal{F}$  should be WR and  $F = 0$  (or  $I - F \in \mathcal{GB}(U)$ ) so that we can normalize  $F$  to zero, as in Lemma 3.7; this is necessarily the case if  $\mathcal{F}$  is UR).

However, the optimal state-feedback for a regular WPLS is not always regular, by Example 11.5 of [WW97], and it is not known whether this holds for all physically relevant systems. Before presenting sufficient conditions to prevent this problem, we state the most general ARE result where we circumvent the problem by dropping (iii) and reformulating (i) of Theorem 6.1. Note that this new equivalence holds for arbitrary (even indefinite and noncoercive) cost functions:

**Theorem 6.2 (Optimal  $K \Leftrightarrow$  ARE)** *Let  $\Sigma$  be WR. Then the following are equivalent:*

- (i) *There is a  $J$ -optimal WR state-feedback operator  $K \in \mathcal{B}(\text{Dom}(A), U)$ ;*
- (ii) *The algebraic Riccati equation (ARE)*

$$K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \quad (38a)$$

$$S = D^*JD + \lim_{s \rightarrow +\infty} B_w^* \mathcal{P}(s - A)^{-1} B, \quad (38b)$$

$$SK = -(B_w^* \mathcal{P} + D^*JC), \quad (38c)$$

has a solution  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ ,  $S = S^* \in \mathcal{B}(U)$ ,  $K \in \mathcal{B}(\text{Dom}(A), U)$  s.t. the feedback  $u(t) = K_w x(t)$  is  $\mathcal{U}_*$ -stabilizing (Definition 6.3).

Moreover, the following hold:

- (a) *Any solution  $\mathcal{P}$  of (ii) is unique (and  $\mathcal{P} = \mathcal{C}_\Sigma^* J \mathcal{C}_\Sigma$ ).*

*The corresponding operators  $K$  in (38) are exactly the WR  $J$ -optimal state-feedback operators over  $\mathcal{U}_*$ . Thus, the minimizing control is then given by the state feedback  $u(t) = K_w x(t)$ , leading to the cost  $\langle x_0, \mathcal{P}x_0 \rangle$ .*

- (b) *There is a WR minimizing state-feedback operator over  $\mathcal{U}_*$  iff (ii) holds and  $\mathcal{J}(0, u) \geq 0$  for all  $u \in \mathcal{U}_*(0)$ .* □

(This follows from Lemma 12.3 and Theorem 10.1(a2)&(b). See Theorem 10.1 for further properties on the solution.)

This motivates us to call a (unique, by (a)) solution  $\mathcal{P}$  of (ii) the  $\mathcal{U}_*$ -stabilizing solution of the ARE:

**Definition 6.3 (ARE)** We call  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$  (or  $(\mathcal{P}, S, K)$ ) a solution of the algebraic Riccati Equation (ARE) (induced by  $\Sigma$  and  $J$ ) iff the ARE (38) is satisfied (with  $K \in \mathcal{B}(\text{Dom}(A), U)$ ,  $S = S^* \in \mathcal{B}(U)$ ).

We call  $\mathcal{P}$  (or  $K$  or  $(\mathcal{P}, S, K)$ ) WR (resp. admissible,  $\mathcal{U}_*$ -stabilizing, ...) if  $\begin{bmatrix} A & B \\ K & 0 \end{bmatrix}$  generates a weakly regular WPLS  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  (resp. and  $[\mathcal{K} \mid \mathcal{F}]$  is admissible,  $\mathcal{U}_*$ -stabilizing, ...). We call  $[\mathcal{K} \mid \mathcal{F}]$   $\mathcal{U}_*$ -stabilizing (with  $\mathcal{P}$ ) if it is admissible,  $\mathcal{K}_\circ x_0 \in \mathcal{U}_*(x_0) \forall x_0 \in H$ , and the following condition (the RCC, residual cost condition) holds:

$$\langle \mathcal{B}^t u + \mathcal{A}_\circ^t x_0, \mathcal{P} \mathcal{A}_\circ^t x_0 \rangle \rightarrow 0, \text{ as } t \rightarrow +\infty \quad (\forall x_0 \in H \quad \forall u \in \mathcal{U}_*(0)). \quad (39)$$

The ARE (38) is given on  $\mathcal{B}(\text{Dom}(A), \text{Dom}(A)^*) \times \mathcal{B}(U) \times \mathcal{B}(\text{Dom}(A), U)$  (just like (36); see Section 9.8 of [M02] for details). If  $\Sigma$  is  $J$ -coercive and  $\mathcal{P}$  is a  $\mathcal{U}_*$ -stabilizing solution, then  $S$  is necessarily invertible (and hence then  $K$  and  $u_{\text{opt}}$  are unique). We shall show in Theorem 9.1(b1) that  $\mathcal{U}_{\text{exp}}$ -stabilizing means exponentially stabilizing. See below Theorem 9.1 for more on  $\mathcal{U}_*$ -stabilizing and the RCC.

As explained in Definition 3.5,  $K$  being a WR state-feedback operator for  $\Sigma$  means that  $\begin{bmatrix} A & B \\ K & 0 \end{bmatrix}$  generate a WR WPLS  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  s.t.  $I - \hat{\mathcal{F}}$  is boundedly invertible on some right half-plane; all this is redundant if, e.g.,  $K \in \mathcal{B}(H, U)$  or if  $B$  and the ARE are as in Corollary 7.5(b)&(c).

Such a  $K$  is  $J$ -optimal if the corresponding closed-loop input  $\mathcal{K}_\circ x_0$  (i.e., the one given by  $u(t) := K_w x(t)$  a.e.) is  $J$ -optimal for each initial state  $x_0 \in H$ . As the sections to follow will reveal, the left column of the closed-loop system  $\Sigma_\circ$  is exactly like  $\Sigma_{\text{opt}}$  of Theorem 4.7 except that it is unique iff  $S$  is one-to-one.

For (38b) and (38c) to be defined, we must have  $\mathcal{P}[H_B] \subset \text{Dom}(B_w^*)$ , where  $\text{Dom}(B_w^*) := \{x_0 \in H \mid \text{w-lim}_{s \rightarrow +\infty} B^* s (s - A^*)^{-1} x_0 \text{ exists}\}$  and  $H_B := (\alpha - A)^{-1} B U + \text{Dom}(A) \subset H$  (this set is independent of  $\alpha \in \rho(A)$ ). By Theorem 6.2, this (and the ARE) is satisfied by the Riccati operator  $\mathcal{P} := \mathcal{C}_\circ^* J \mathcal{C}_\circ$  when there is a WR  $J$ -optimal state feedback  $\mathcal{F}$  (with no feedthrough) and  $\mathcal{D}$  is WR. See Remark 9.8.3 of [M02] for further details on, e.g.,  $K$  satisfying the above requirements, and the rest of Chapter 9 for simplifications of the equation and for further results.

Under certain additional smoothness, any unique optimal control is given by regular state feedback, and in some cases we even have  $S = D^* J D$  and  $B_w^* \mathcal{P} \in \mathcal{B}(H, U)$ , as in Theorem 6.7 below. For general regular systems, the Riccati operator need not satisfy  $\mathcal{P}[H] \subset \text{Dom}(B_w^*)$ , not even if there is a WR  $J$ -optimal state-feedback operator (see, e.g., Example 9.13.8 of [M02]), and we do not know a priori whether an optimal control is even well-posed (by Example 8.4.13 of [M02]; cf. the difference between “1.” and “2.” on p. 67).

As in the discrete-time case (where  $S = D^* J D + B^* \mathcal{P} B$ ), the definiteness of the indicator or *signature operator*  $S$  is inherited from the Popov Toeplitz operator  $\mathcal{S}_{\text{PT}}$  or  $\mathcal{J}(0, \cdot)$ , i.e., from the underlying optimal control problem (this is not true for  $D^* J D$ !). See p. 35 for details. Moreover, the cost becomes  $\langle y, J y \rangle = \langle x_0, \mathcal{P} x_0 \rangle_H + \langle u_\circ, S u_\circ \rangle$  if we add an external input  $u_\circ \in L_c^2(\mathbb{R}_+; U)$  to the optimally controlled closed-loop system. (In fact, this paragraph is true even if  $D$  does not exist (i.e., if  $\mathcal{D}$  is irregular); see Section 10 for details.)

Next we show that the RCC is not redundant for  $\mathcal{U}_{\text{out}}$  (otherwise  $\mathcal{P} = 2$  would be  $\mathcal{U}_{\text{out}}$ -stabilizing, hence  $K = -2$  would be  $J$ -optimal over  $\mathcal{U}_{\text{out}}$ ):

**Example 6.4** (RCC; exp. stabilizing cannot be q.r.c.). Let  $\Sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $J = 1$ . Obviously,  $\mathcal{J}(x_0, u) = \|u\|_2^2$ , hence  $K = 0$  is the unique  $J$ -optimal state-feedback operator over  $\mathcal{U}_{\text{out}}$ . The  $\mathcal{U}_{\text{out}}$ -stabilizing solution  $\mathcal{P} = 0$  (“no feedback needed to minimize  $\|y\|_2^2$  over  $\mathcal{U}_{\text{out}}$ ” because  $\Sigma$  is already output-stable) of the ARE  $(-\mathcal{P})^2 = 1\mathcal{P} + \mathcal{P}1 + 0$ ,  $S = 1$ ,  $K = -\mathcal{P}$  differs from the  $\mathcal{U}_{\text{exp}}$ -stabilizing solution  $\mathcal{P} = 2$  (“feedback  $u(t) = -2x(t)$  (leading to cost  $2|x_0|^2$ ) needed to minimize  $\|y\|_2^2$  over  $\mathcal{U}_{\text{exp}}$ ”).

Trivially,  $1 = 1 \cdot 1^{-1}$  is a q.r.c.f. of  $\mathcal{D} = 1$ . A coprime stabilization (such as the zero feedback above) means (in the finite-dimensional case) that “ $\hat{\mathcal{N}}$  and  $\hat{\mathcal{M}}$  have no common zeros on  $\overline{\mathbb{C}^+}$ ”, i.e., that one stabilizes as little as possible (only the poles of  $\hat{\mathcal{D}}$ ). The semigroup  $A = 1$  has more poles (namely  $s = 1$ ) than the transfer function  $\hat{\mathcal{D}} = 1$  (which has none), hence one must introduce additional zeros to  $\hat{\mathcal{M}}$  (and hence to  $\hat{\mathcal{N}} = \widehat{\mathcal{D}_\circ} = \hat{\mathcal{D}} \hat{\mathcal{M}}$  too:  $\hat{\mathcal{M}}(1) = 0 = \hat{\mathcal{N}}(1)$ ) to stabilize the semigroup too ( $\mathcal{U}_{\text{exp}}$  vs.

$\mathcal{U}_{\text{out}}$ ). Thus, no exponentially stabilizing state-feedback for the system  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  can be q.r.c.-stabilizing.  $\triangleleft$

(See Example 9.13.2 of [M02] for further details.)

If  $B$  is not maximally unbounded, then any state-feedback is UR, hence then Theorem 5.1 implies that, for any positively  $J$ -coercive cost function, the FCC holds iff there is a UR minimizing state-feedback operator (see (v)):

**Lemma 6.5** *Assume that  $B$  is not maximally unbounded, i.e., that there are  $M, R, \epsilon > 0$  s.t.  $\|(s - A)^{-1}B\|_{\mathcal{B}(U,H)} \leq Ms^{-\frac{1}{2}-\epsilon}$  for  $s \in (R, \infty)$ . Then  $\Sigma$  is uniformly regular (UR). Consequently, the following are equivalent*

- (i) *There is a  $J$ -optimal state-feedback pair over  $\mathcal{U}_*$ .*
- (ii) *There is a UR  $J$ -optimal state-feedback operator over  $\mathcal{U}_*$ .*
- (iii) *The IRE has a  $\mathcal{U}_*$ -stabilizing solution.<sup>6</sup>*
- (iv) *The ARE has a  $\mathcal{U}_*$ -stabilizing solution.*

Moreover, the w-lim in the ARE converges uniformly to zero and the optimal UR state-feedback is given by  $u(t) = K_w x(t)$  a.e., with cost  $\langle x_0, \mathcal{P}x_0 \rangle$ .

For positively  $J$ -coercive problems (having  $\vartheta = 0$ ), a fifth equivalent condition is

- (v) *The FCC holds.*

(The proof is given on p. 63.)

(The inequality can always be established for  $\epsilon = 0$ ; for  $\epsilon = 1/2$  it holds iff  $B$  is bounded (in which case (i)–(v) are equivalent for any  $J$ -coercive problems and the w-lim condition becomes redundant, by Theorem 6.7). A sufficient condition is that  $A$  is analytic and  $(s_0 - A)^{-\beta}B$  is bounded for some  $\beta < 1/2$ ,  $s_0 \in \rho(A)$ , by Lemma 9.4.2(k) of [M02].)

It follows that in the results of Section 5, when  $B$  is not maximally unbounded, the stabilizability condition (or FCC) is equivalent to the solvability of the corresponding ARE, whose solution provides the desired (UR) stabilizing state-feedback operator, as explained in Corollary 7.5(b)&(c).

We now apply the above equivalence of (i)–(v) to a detectable LQR problem, so that “ $\mathcal{U}_*$ -stabilizing” can be ignored (as long as  $\mathcal{P} \geq 0$ ):

**Corollary 6.6 (LQR,  $B$ )** (a) *Assume that  $B$  is not maximally unbounded, and let  $R, T \gg 0$ ,  $Q \geq 0$ . Then, for each initial state  $x_0 \in H$ , there is a control  $u \in L^2(\mathbb{R}_+; U)$  s.t. the cost*

$$\mathcal{J}(x_0, u) := \int_0^\infty (\langle y, Qy \rangle_Y + \langle x, Tx \rangle_H + \langle u, Ru \rangle_U) dm, \quad (40)$$

*is finite iff the ARE*

$$K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*QC + T, \quad (41a)$$

$$S = D^*QD + R, \quad (41b)$$

$$SK = -(B_w^*\mathcal{P} + D^*QC), \quad (41c)$$

*has a nonnegative solution  $\mathcal{P} \in \mathcal{B}(H)$  satisfying  $\lim_{s \rightarrow +\infty} B_w^*\mathcal{P}(s - A)^{-1}B = 0$ .*

*Assume that  $(\mathcal{P}, S, K)$  is such a solution. Then  $K$  is the unique uniformly regular  $J$ -optimal state-feedback operator, and it is exponentially stabilizing and leads to the minimal cost, which equals  $\langle x_0, \mathcal{P}x_0 \rangle$ .*

(b) *Instead of  $T \gg 0$ , assume that  $T \geq 0$  and  $Q \gg 0$ . Then everything in (a) still holds except that “Then  $K \dots$ ” holds for the smallest nonnegative solution  $\mathcal{P}$  only (which exists whenever there are any solutions, equivalently, whenever the FCC holds),  $K$  is SOS-stabilizing and  $\mathcal{N}, \mathcal{M}$  become q.r.c.*

(The proof is given on p. 65. Note that Theorems 1.1 and 1.3 are special cases of this and that  $\mathcal{P}$  is unique in (a), being the  $J$ -optimal cost operator.)

The above FCC “ $\forall x_0 \exists u \in L^2$  s.t.  $\mathcal{J} < \infty$ ” is obviously equivalent to the state-FCC (3) in (a) (to the output-FCC in (b) if  $T = 0$ ).

<sup>6</sup>See Definition 7.3.



When can one remove the above w-lim condition? If  $H_B \subset Z \subset H$  continuously, and  $Z$  is a Banach space with  $(s - A)^{-1}B \rightarrow 0$  in  $Z$  as  $s \rightarrow +\infty$  (this is true for  $Z = H$ ), then  $\mathcal{P}[Z] \subset \text{Dom}(B_w^*)$  is a sufficient condition; this also applies to indefinite problems. In Section 9.4 of [M02] we give sufficient conditions in the case of an analytic semigroups; below we study the case  $Z = H$ .

Under certain assumptions, the ARE becomes equivalent to the following conditions (the  $B_w^*$ -ARE):  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ ,  $\mathcal{P}[H] \subset \text{Dom}(B_w^*)$ , and

$$(B_w^* \mathcal{P} + D^* J C)^* (D^* J D)^{-1} (B_w^* \mathcal{P} + D^* J C) = A^* \mathcal{P} + \mathcal{P} A + C^* J C. \quad (42)$$

Moreover, then a unique  $J$ -optimal control is necessarily given by an ULR state-feedback operator:

**Theorem 6.7 ( $B_w^*$ -ARE  $\Leftrightarrow J$ -optimal)** *Assume that at least one of (1.)–(4.) below holds:*

- (1.)  $B$  is bounded (i.e.,  $B \in \mathcal{B}(U, H)$ );
- (2.)  $\mathcal{A}B \in L^1([0, 1]; \mathcal{B}(U, H))$  and  $C \in \mathcal{B}(H, Y)$ ;
- (3.)  $\mathcal{A}Bu_0 \in L^2([0, 1]; H)$  and  $C_w \mathcal{A}Bu_0 \in L^2([0, 1]; Y)$  for all  $u_0 \in U$ ;
- (4.) (**Stable case**)  $C \in \mathcal{B}(H, Y)$ ,  $D^* J C = 0$ ,  $\mathcal{D} \in \mathcal{B}(U, Y) + \mathcal{B}(U, L^1(\mathbb{R}_+; Y))^*$ , and  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  and  $\mathcal{B}\tau$  is stable (or  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  and  $\mathcal{C}$  is stable).

Then  $\mathcal{D}$  is ULR. If  $D^* J D \in \mathcal{GB}(U)$ , then the following are equivalent:

- (i) There is a unique  $J$ -optimal control over  $\mathcal{U}_*(x_0)$  for each  $x_0 \in H$ .
- (ii) There is a  $J$ -optimal state-feedback pair over  $\mathcal{U}_*$ .
- (iii) The IRE or the ARE or the  $B_w^*$ -ARE has a  $\mathcal{U}_*$ -stabilizing solution.

If (iii) holds, then the IRE, ARE and  $B_w^*$ -ARE have the same  $\mathcal{U}_*$ -stabilizing solution (with  $S = D^* J D$ ), hence then Theorem 10.1 applies; moreover, then  $K := -(D^* J D)^{-1} (B_w^* \mathcal{P} + D^* J C)$  is the unique ULR  $J$ -optimal state-feedback operator.  $\square$

(This follows from Theorems 9.2.9 and 9.2.3 of [M02]; in the same section also further alternatives for (1.)–(4.) and numerous further results are given. As an example, if  $\mathcal{A}Bu_0 \in L^1([0, 1]; H) \forall u_0 \in U$ , then the state-FCC holds iff there is  $\mathcal{P} \geq 0$  s.t.  $\mathcal{P}[H] \subset \text{Dom}(B_w^*)$  and  $(B_w^* \mathcal{P})^* B_w^* \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + I$ . Moreover, then  $K := -B_w^* \mathcal{P} \in \mathcal{B}(H, U)$  is ULR and exponentially stabilizing.)

In contrast to Lemma 6.5, we note that here (a) we do not need positive  $J$ -coercivity to guarantee the existence of  $K$  (although  $J$ -coercivity and the FCC is sufficient for (i), by Theorem 4.6), in particular, also the indefinite case is covered; (b) the condition w-lim  $B_w^* \mathcal{P}(s - A)^{-1}B = 0$  is replaced by the stronger assumption that  $\mathcal{P}[H] \subset \text{Dom}(B_w^*)$ , or equivalently, w-lim $_{s \rightarrow +\infty} B^* s(s - A)^{-1} \mathcal{P} x_0 \forall x_0 \in H$  must exist for all  $x_0 \in H$ .

**Notes for Section 6:** The necessity of equations (38) for SR stable  $J$ -coercive problems over  $\mathcal{U}_{\text{out}}$  was shown by Olof Staffans [S98b] (see Remark 5.2 of [S98c]). At the same time, (38a) and (38c) were discovered independently by Martin Weiss and George Weiss [WW97]. In the same setting, we proved the sufficiency in [M97].

The above (new) frequency-domain proof for Theorem 6.2 is significantly shorter and simpler than our original time-domain proof of [M02] (Section 9.11). However, the latter, technically more demanding but closer to finite-dimensional ones, can more easily be generalized to finite-horizon, time-variant and/or nonlinear settings.

A number of further results, special cases and notes are given in Chapters 9–10 of [M02] (see, e.g., Section 10.1 for LQR results), including Riccati inequalities and relations to spectral and coprime factorizations. Corresponding results on discrete-time AREs are presented in Chapter 14 of [M02]. See Section 9.13 of [M02] for examples where, e.g.,  $\mathcal{D}$  and  $\mathcal{F}$  are regular but  $\mathcal{P}[H] \not\subset \text{Dom}(B_w^*)$  (although  $\mathcal{P}[H_B] \subset \text{Dom}(B_w^*)$ ) or where  $\mathcal{D}$  is very regular but  $\mathcal{F}$  not regular at all.

Under mild assumptions, a minimizing state-feedback operator also solves the “ $H^2$  problem” (see Section 10.4 of [M02] for definition and proofs).

The fact that  $\Sigma$  is UR when  $B$  is not maximally unbounded is due to G. Weiss [WC99], who applied it to the stable LQR problem. For exponentially detectable systems with analytic semigroups, the results in [LT00] allow for significantly more unbounded  $B$ ’s

than Corollary 6.6 does (they have the corresponding indefinite result too, both for highly coercive cost functions). However, there do not seem to exist similar results for non-analytic semigroups, and Lemma 6.5 covers more general cost functions. Further optimization and ARE results for as general cost functions can be found in [LR95] and [IOW99], for finite-dimensional systems.

For Pritchard–Salamon systems<sup>7</sup> that are smooth, most of Theorem 6.1 was proved in [vK93], Theorem 3.10. Theorem 6.7(1.) extends those results. See also Theorem 11.2.

## 7 Integral Riccati equations (IREs) and optimal control

By Theorem 4.7, a unique optimal control can always be given in WPLS form (i.e., as a “generalized state feedback”, see Definition 3.2). Traditionally, this control is determined by finding the stabilizing solution of the corresponding (infinitesimal) algebraic Riccati equation (ARE); this was illustrated in the previous section.

However, without significant regularity assumptions, such as those above, the feedthrough operator (often normalized to  $F = 0$ , as above) of the optimal state-feedback loop (“ $u(t) = K_w x(t) + Fu(t)$  for a.e.  $t \geq 0$ ”) need not exist. In fact, sometimes this loop is even ill-posed! Nevertheless, we can use certain integral Riccati equations (IREs) to characterize the optimal control.

In Theorem 7.1 we shall show that a unique optimal control is the one given by the  $\mathcal{U}_*$ -stabilizing solution of the  $\mathcal{S}^t$ -IRE (or  $\hat{\mathcal{S}}^t$ -IRE). The “generalized state-feedback loop” ( $u(t) = K_w x(t)$  a.e.  $t \geq 0$ ) of this control is well posed iff the IRE has a  $\mathcal{U}_*$ -stabilizing solution (equivalently, any of (i)–(vi) of Theorem 7.2 holds). We reduce this condition to a stable spectral factorization problem (Theorem 7.2(iv)).

These results form a direct generalization of the classical (algebraic) RE theory to an extent that cannot be covered by the (standard) ARE. In addition, they will be used to prove the stabilization and factorization results of Section 5.

We start by noting that a control  $\mathcal{K}_{\text{opt}}$  in WPLS form is optimal and  $\mathcal{P}$  is the optimal cost operator iff  $\mathcal{P}, \mathcal{K}_{\text{opt}}$  satisfy the  $\mathcal{S}^t$ -IRE:

**Theorem 7.1 ( $\mathcal{S}^t$ -IRE &  $\hat{\mathcal{S}}^t$ -IRE)** *Let  $\mathcal{K}_{\text{opt}}$  be a control in WPLS form for  $\Sigma$ , and let  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ ,  $\omega \geq \max\{\omega_A, \omega_{A_{\text{opt}}}\}$ .*

*Then  $\mathcal{K}_{\text{opt}}x_0$  is  $J$ -optimal and  $\mathcal{P}$  is its Riccati operator  $\mathcal{C}_{\text{opt}}^* J \mathcal{C}_{\text{opt}}$  iff  $\mathcal{P}, \mathcal{K}_{\text{opt}}$  is a  $\mathcal{U}_*$ -stabilizing solution of the following equations (the  $\mathcal{S}^t$ -IRE) for all  $t \geq 0$ :*

$$\mathcal{K}_{\text{opt}}^t \mathcal{S}^t \mathcal{K}_{\text{opt}}^t = \mathcal{A}^{t*} \mathcal{P} \mathcal{A}^t - \mathcal{P} + \mathcal{C}^{t*} J \mathcal{C}^t, \quad (43a)$$

$$\mathcal{S}^t := \mathcal{D}^{t*} J \mathcal{D}^t + \mathcal{B}^{t*} \mathcal{P} \mathcal{B}^t, \quad (43b)$$

$$\mathcal{S}^t \mathcal{K}_{\text{opt}}^t = - \left( \mathcal{D}^{t*} J \mathcal{C}^t + \mathcal{B}^{t*} \mathcal{P} \mathcal{A}^t \right) \quad (43c)$$

Moreover, equations (43) hold iff the following equations (the  $\hat{\mathcal{S}}^t$ -IRE) hold for some (equivalently, all)  $s, z \in \mathbb{C}_\omega^+$ :

$$\widehat{\mathcal{K}_{\text{opt}}}(s)^* \hat{\mathcal{S}}(s, z) \widehat{\mathcal{K}_{\text{opt}}}(z) = (s - A)^{-*} (A^* \mathcal{P} + \mathcal{P} A + C^* J C) (z - A)^{-1}, \quad (44a)$$

$$\hat{\mathcal{S}}(s, z) := \hat{\mathcal{D}}(s)^* J \hat{\mathcal{D}}(z) + (z + \bar{s}) B^* (s - A)^{-*} \mathcal{P} (z - A)^{-1} B, \quad (44b)$$

$$\hat{\mathcal{S}}(s, z) \widehat{\mathcal{K}_{\text{opt}}}(z) = - \hat{\mathcal{D}}(s)^* J C (z - A)^{-1} - B^* (s - A)^{-*} \mathcal{P} (s^* + A) (z - A)^{-1}. \quad (44c)$$

(This follows from Lemma 9.6 and Theorem 9.1.)

By  $\mathcal{U}_*$ -stabilizing we mean that  $\mathcal{K}_{\text{opt}}x_0 \in \mathcal{U}_*(x_0) \forall x_0 \in H$  and the RCC (39) holds (with  $\mathcal{A}_{\text{opt}}$  in place of  $\mathcal{A}_\diamond$ ). By Theorem 9.1(b1),  $\mathcal{U}_{\text{exp}}$ -stabilizing is equivalent to “ $\Sigma_{\text{opt}}$  is exponentially stable” (equivalently, to  $\mathcal{A}_{\text{opt}}x_0 \in L^2(\mathbb{R}_+; H) \forall x_0 \in H$ ).

Note that we have  $\mathcal{K}_{\text{opt}}x_0 \in L_\omega^2(\mathbb{R}_+; U)$  for some  $\omega \in \mathbb{R}$  (Definition 2.1), hence some (unique) holomorphic  $\widehat{\mathcal{K}_{\text{opt}}}: \mathbb{C}_\omega^+ \rightarrow \mathcal{B}(H, U)$  satisfies  $\widehat{\mathcal{K}_{\text{opt}}}(s) = \widehat{\mathcal{K}_{\text{opt}}}(s)x_0$  on  $\mathbb{C}_\omega^+$  for all  $x_0 \in H$ .

<sup>7</sup>P–S systems are exactly the WPLS with a bounded input operator ( $B$ ) that can be written as WPLS with a bounded output operator ( $C$ ) by changing the state space, as shown in [M02], Lemma 6.9.4.

For a fixed  $t > 0$ , the  $\mathcal{S}^t$ -IRE (43) coincides with the (discrete-time) algebraic Riccati equation for the discretized system  $\begin{bmatrix} \mathcal{A}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{D}^t \end{bmatrix}$ ; this fact provides an alternative proof for the theorem (see Theorem 14.1.6 and Proposition 9.8.7 of [M02]; it also follows that “all  $t \geq 0$ ” is equivalent to “some  $t > 0$ ”).

Thus, given the FCC and  $J$ -coercivity ( $\mathcal{S}_{PT} \in \mathcal{GB}$ ), there is a unique optimal control, it is given in the WPLS form (i.e., as generalized state feedback, by Theorems 4.6 and 4.7), and it satisfies the  $\mathcal{S}^t$ -IRE and the  $\mathcal{S}$ -IRE. But is it given by (well-posed) state-feedback?

The answer is “not always” (unless  $\mathcal{S}_{PT} \gg 0$  or the system is rather smooth), by Example 8.4.13 of [M02]. The answer is positive iff the spectral factorization problem (iv) below has a solution, equivalently, iff the (optimally truncated Popov Toeplitz) operator  $\mathcal{S}^t$  can be factorized as  $\mathcal{X}^{t*} S \mathcal{X}^t$ , again equivalently, iff  $\hat{\mathcal{S}}(s, s)$  can be factorized as  $\hat{\mathcal{X}}(s)^* S \hat{\mathcal{X}}(s)$ :

**Theorem 7.2** ( $\hat{\mathcal{S}} = \hat{\mathcal{X}}^* S \hat{\mathcal{X}} \Leftrightarrow \exists [\mathcal{K} \mid \mathcal{F}]$ ) Assume that there is a unique  $J$ -optimal control for each  $x_0 \in H$ . Define  $\mathcal{P}$  and  $\mathcal{S}^t$  as in Theorem 7.1. Then the following are equivalent:

- (i) There is a  $J$ -optimal state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$ .
- (ii) There are  $\hat{\mathcal{X}} \in \mathcal{GH}_\infty(U)$ ,  $S \in \mathcal{B}(U)$  s.t.

$$\hat{\mathcal{X}}(s)^* S \hat{\mathcal{X}}(s) = \hat{\mathcal{D}}(s)^* J \hat{\mathcal{D}}(s) + 2 \operatorname{Re} s B^* (s - A)^{-*} \mathcal{P} (s - A)^{-1} B \quad (45)$$

on some right half-plane (equivalently, on a strip  $\mathbb{C}_\alpha^+ \setminus \mathbb{C}_\beta^+$ , where  $\omega_A \leq \alpha < \beta < \infty$ ).

- (iii) There are  $\mathcal{X} \in \mathcal{GTIC}_\infty(U)$ ,  $S \in \mathcal{B}(U)$  that satisfy  $\mathcal{X}^{t*} S \mathcal{X}^t = \mathcal{S}^t \forall t > 0$ .
- (iv) There are  $\mathcal{X}_+ \in \mathcal{GTIC}(U)$ ,  $S = S^* \in \mathcal{B}(U)$  s.t.  $S$  is one-to-one and  $\mathcal{X}_+^* S \mathcal{X}_+ = \mathcal{D}_+^* J_+ \mathcal{D}_+$  for some  $\alpha > \max\{0, \omega_A\}$ .

Here  $\mathcal{D}_+ := \begin{bmatrix} e^{-\alpha} \mathcal{D} e^{\alpha} \\ e^{-\alpha} \mathcal{D}_T e^{\alpha} \end{bmatrix} \in \mathcal{TIC}_{-\delta}$  for some  $\delta > 0$  and  $J_+ := \begin{bmatrix} J & 0 \\ 0 & 2\alpha \mathcal{P} \end{bmatrix}$ .

- (v) The IRE (46) has a  $\mathcal{U}_*$ -stabilizing solution.
- (vi) The  $\widehat{\text{IRE}}$  (47) has a  $\mathcal{U}_*$ -stabilizing solution.
- (vii) There is an admissible state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$  s.t.  $\mathcal{K}_{\text{opt}} = \mathcal{K}_\odot$ .

Moreover, the following hold:

- (a) The solutions of (i)–(vii) are equal (with  $\mathcal{F} = I - \mathcal{X}$ ,  $\mathcal{K} = \mathcal{X} \mathcal{K}_{\text{opt}}$ ,  $\mathcal{X}_+ := e^{-\alpha} \mathcal{X} e^{\alpha}$ ).
- (b) Given one solution  $(\mathcal{X}, S)$ , all solutions are given by  $(E \mathcal{X}, E^{-*} S E^{-1})$  ( $E \in \mathcal{GB}(U)$ ), and the operator  $S$  is one-to-one. If  $\mathcal{S}_{PT}$  is invertible, then so is  $S$ . Also the rest of Theorem 10.1 applies.  $\square$

(The proof is given by Lemma 10.7. See (138) for  $\mathcal{K}$  (and  $\mathcal{K}_\odot$ ) in terms of  $\mathcal{X}$ ,  $S$ ,  $\mathcal{P}$ ,  $\Sigma$  and  $J$ .) If  $\mathcal{U}_*$  equals  $\mathcal{U}_{\text{out}}$  or  $\mathcal{U}_{\text{exp}}$ , then one more equivalent condition is that  $\mathcal{D}$  has a “ $J$ -optimal factorization” (a generalization of spectral factorization), by Theorem 9.14.3 of [M02]. Another equivalent condition is that  $s \hat{\mathcal{K}}(s) B_-$ ,  $s \hat{\mathcal{K}}_\odot(s) B_- \in H_\infty$  (as functions  $s \rightarrow \mathcal{B}(U)$ ), as will be shown in [M03b]; here  $B_- := A^{-1} B \in \mathcal{B}(U, H)$  and  $\hat{\mathcal{K}}(s) = K(s - A)^{-1}$ , where  $K$  is determined by the so called reciprocal ARE.

The spectral factorization condition (iv) seems independent of  $\mathcal{U}_*$ . Of course, that cannot be the case: the information on  $\mathcal{U}_*$  is carried by  $\mathcal{P}$ .

When  $\mathcal{S}_{PT} \gg 0$  (and, e.g.,  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  or  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$ ), we can show that condition (iv) can be satisfied whenever the FCC holds (p. 62). This will establish Theorem 5.1 and the other results presented Section 5.

The *signature operator* (indicator)  $S$  has obviously the same definiteness as  $\mathcal{S}^t$ , which in turn inherits (a restriction of) that of the Popov Toeplitz operator  $\mathcal{S}_{PT}$ , as noted in Lemma 9.8. In particular,  $\mathcal{S}_{PT} \geq 0$  (resp.  $> 0$ ,  $\gg 0$ ,  $\in \mathcal{GB}$ , is one-to-one)  $\Rightarrow S \geq 0$  (resp.  $> 0$ ,  $\gg 0$ ,  $\in \mathcal{GB}$ , is one-to-one). In fact, at least if  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  (or  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  and  $\mathcal{N}, \mathcal{M}$  are q.r.c.), then also the converse implications hold and  $\mathcal{X}u \in L^2$  &  $\mathcal{J}(0, u) = \langle \mathcal{X}u, S \mathcal{X}u \rangle = \langle u, \mathcal{S}^t u \rangle \forall u \in \mathcal{U}_*(0)$ . See Theorem 9.9.1(f2)&(h)&(k), Lemma 9.10.3, Theorem 8.4.5(d) and pp. 482&387 of [M02] for details.

By Theorem 7.2, an admissible state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$  for  $\Sigma$  is  $J$ -optimal iff it is a  $\mathcal{U}_*$ -stabilizing solution of the IRE (with some  $\mathcal{P}, S$ ):

**Definition 7.3 (A  $\mathcal{U}_*$ -stabilizing solution of the IRE (or  $\widehat{\text{IRE}}$ )**  $((\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}]), \mathcal{X}, \mathcal{M}, \mathcal{N}, \Sigma_\odot))$   
We call  $\mathcal{P}$  (or  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$ ) a solution of the Integral Riccati Equation (IRE) (induced by  $\Sigma$  and  $J$ ) iff the IRE

$$\mathcal{K}^{t*} S \mathcal{K}^t = \mathcal{A}^{t*} \mathcal{P} \mathcal{A}^t - \mathcal{P} + \mathcal{C}^{t*} J \mathcal{C}^t, \quad (46a)$$

$$\mathcal{X}^{t*} S \mathcal{X}^t = \mathcal{D}^{t*} J \mathcal{D}^t + \mathcal{B}^{t*} \mathcal{P} \mathcal{B}^t, \quad (46b)$$

$$\mathcal{X}^{t*} S \mathcal{K}^t = - \left( \mathcal{D}^{t*} J \mathcal{C}^t + \mathcal{B}^{t*} \mathcal{P} \mathcal{A}^t \right) \quad (46c)$$

(here  $\mathcal{X} := I - \mathcal{F}$ ) is satisfied for all  $t > 0$ , and  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ ,  $S = S^* \in \mathcal{B}(U)$ ,  $\mathcal{K} \in \mathcal{B}(H, \mathcal{L}_{\text{loc}}^2(\mathbb{R}_+; U))$ , and  $\mathcal{F} \in \text{TIC}_\infty(U)$ .

We call  $\mathcal{P}$  admissible or  $\mathcal{U}_*$ -stabilizing if  $[\mathcal{K} \mid \mathcal{F}]$  is (see Definition 6.3).

Solutions of the  $\widehat{\text{IRE}}$  are defined in the same way, except that instead of (46) we require that  $[\frac{\mathcal{A}}{\mathcal{X}} \mid \frac{\mathcal{C}}{\mathcal{F}}]$  is a WPLS and that the following  $\widehat{\text{IRE}}$  is satisfied for some  $s = z \in \mathbb{C}_{\omega_A}^+$ :

$$K^* S K = A^* \mathcal{P} + \mathcal{P} A + C^* J C, \quad (47a)$$

$$\hat{\mathcal{X}}(s)^* S \hat{\mathcal{X}}(z) = \hat{\mathcal{D}}(s)^* J \hat{\mathcal{D}}(z) + (z + \bar{s}) B^* (s - A)^{-*} \mathcal{P} (z - A)^{-1} B, \quad (47b)$$

$$\hat{\mathcal{X}}(s)^* S K (z - A)^{-1} = -\hat{\mathcal{D}}(s)^* J C (z - A)^{-1} - B^* (s - A)^{-*} \mathcal{P} (s^* + A) (z - A)^{-1}. \quad (47c)$$

(By Lemma 10.2, this implies that (47) actually holds for all  $s, z \in \rho(A)$ . Note from the definition that we only study the self-adjoint solutions.)

As in Definition 3.5, for admissible  $\mathcal{P}$ , we denote the corresponding closed-loop system by  $\Sigma_\odot$  and set  $\mathcal{X} := I - \mathcal{F} \in \text{TIC}_\infty(U)$ ,  $\mathcal{M} := \mathcal{X}^{-1} \in \mathcal{GTIC}_\infty(U)$ ,  $\mathcal{N} := \mathcal{D}_\odot := \mathcal{D} \mathcal{M} \in \text{TIC}_\infty(U, Y)$ .

It suffices to require (46) for some  $t > 0$  in Theorem 7.2(v), by the comments below Theorem 9.1.

From Theorem 6.2 (or Definition 6.3) we observe that any admissible (resp.  $\mathcal{U}_*$ -stabilizing) solution of the ARE is an admissible (resp.  $\mathcal{U}_*$ -stabilizing) solution of the IRE (the converse holds iff  $\mathcal{D}, \mathcal{F}$  are WR and  $F = 0$ ).

If  $B$  is bounded,  $C = \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix}$ ,  $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$ ,  $J = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$  (hence  $\mathcal{J}(x_0, u) = \|u\|_2^2 + \|\tilde{C}x\|_2^2$ ), then, by (41), we get  $S = I$ ,  $K = -B^* \mathcal{P}$ , hence then the ARE reduces to  $\mathcal{P} B B^* \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + \tilde{C}^* \tilde{C}$ . This ARE is equivalent to (46a), which becomes

$$\mathcal{P} x_0 = \mathcal{A}^{t*} \mathcal{P} \mathcal{A}^t x_0 + \int_0^t \mathcal{A}^{s*} (\tilde{C}^* \tilde{C} - \mathcal{P} B B^* \mathcal{P}) \mathcal{A}^s x_0 ds \quad \forall x_0 \in H, \quad (48)$$

familiar from classical results, such as equation (4.26) of [G79] (take  $s = 0$ ,  $D = \tilde{C}^* \tilde{C}$ ,  $Q = I$ ).

If  $B$  is bounded and  $D^* J D$  is invertible, then all the above REs (and the ones presented in Theorem 9.1) are equivalent, and it suffices to verify the equations (since every solution generates an admissible state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$ ):

**Lemma 7.4 (Bounded  $B$ : ARE  $\Leftrightarrow \mathcal{S}^t$ -IRE)** Assume that  $B \in \mathcal{B}(U, H)$ .

(a) If  $(\mathcal{P}, S, K)$  is a WR solution of the ARE, then it is admissible and ULR and  $(\mathcal{P}, \mathcal{K}_\odot)$  solve the  $\mathcal{S}^t$ -IRE,  $\hat{\mathcal{S}}$ -IRE,  $\widehat{\text{IRE}}$  and IRE. If  $D^* J D \in \mathcal{GB}(U)$ , then any solution of the ARE is WR.

(b) Conversely, if  $\mathcal{K}_{\text{opt}}$  is a control in WPLS form and  $(\mathcal{P}, \mathcal{K}_{\text{opt}})$  solve the  $\mathcal{S}^t$ -IRE or the  $\hat{\mathcal{S}}$ -IRE, then  $\mathcal{K}_{\text{opt}} = \mathcal{K}_\odot$  for some  $K$  which is as in (a).

(The proof is given on p. 61.)

Most results of Section 5 provide equivalent conditions for the existence of a certain kind of stabilizing state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$ . By (c), one more equivalent condition is that the IRE has a nonnegative admissible solution:

**Corollary 7.5 ( $\mathcal{P} \geq 0 \Leftrightarrow \text{minimizing}$ )** (a) In any of the results mentioned in Corollary 8.3(a)/(b) below, one more equivalent condition is that the corresponding IRE[s] (equivalently,  $\mathcal{S}^t$ -IRE[s]) has a  $\mathcal{U}_*$ -stabilizing solution.

(b) If  $B$  [and  $C^*$ ] is not maximally unbounded, then a further equivalent condition is that the corresponding ARE[s] has a  $\mathcal{U}_*$ -stabilizing solution. Moreover,  $S = D^* J D \gg 0$ ,

and we can have  $[\mathcal{K} \mid \mathcal{F}]$  generated by  $[K \mid 0]$ , where  $K$  is from the ARE [and  $[\frac{\mathcal{H}}{\mathcal{G}}]$  by  $[\frac{H}{0}]$ ,  $H = \tilde{K}^*$ , where  $(\tilde{P}, \tilde{S}, \tilde{K})$  is the solution of the dual ARE].

(c) Exclude Theorem 5.1 from the results mentioned above. Then the existence of an admissible nonnegative solution[s] of the IRE[s] (or any nonnegative solution[s] of the ARE[s] in (b)) is another equivalent condition. Moreover, any nonnegative UR solution of the ARE is admissible; if  $\dim U < \infty$  [ $\dim Y < \infty$  for the dual IREs] or  $B$  [and  $C^*$ ] is not maximally unbounded, then any nonnegative solution of the ARE is admissible. Any admissible solution is SOS-stabilizing.

(A nonnegative admissible solution of the state-IRE (or of the state-ARE) is  $\mathcal{U}_{\text{exp}}$ -stabilizing (hence unique); the  $\mathcal{U}_{\text{out}}$ -stabilizing solution of the output-IRE (or output-ARE), if any, is the smallest admissible nonnegative solution.)

Naturally, “ $C^*$  not maximally unbounded” means that  $\|(s - A^*)^{-1}C^*\|_{\mathcal{B}(Y, H)} \leq Ms^{-\frac{1}{2}-\epsilon}$  for  $s \in (R, \infty)$  and some  $R, M < \infty$ . See Corollary 8.3(c) and Remark 8.4 for (b) and (c) under alternative assumptions on  $\Sigma$ . Similar claims also hold for the  $B_w^*$ -ARE, since any of its solutions is an admissible solution of the IRE, by Proposition 9.2.7 of [M02]. If  $\mathcal{D}$  is UR, then a solution of the ARE is UR iff the limit in the ARE converges uniformly (this is the case in most applications), by Lemma 9.11.5(e) of [M02].

Here  $\mathcal{U}_*$  and the IRE should be the same as in the proof of that result (hence  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  and  $J = I$  except possibly for Theorem 5.1; moreover,  $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$  for, e.g., Corollary 5.2). Thus, the IRE or ARE is determined by the system (sometimes “ $\tilde{\Sigma}$ ” instead of  $\Sigma$ ) and the  $J$  used in the proof. Note that, e.g., in the proof of Corollary 5.2 we have  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  and  $[\mathcal{C} \mid \mathcal{D}] = [\mathcal{C} \mid \mathcal{D}]$  (i.e.,  $(C \mid D) = (\mathcal{C} \mid \mathcal{D})$ ), hence the ARE in (b) becomes the *state-FCC ARE*

$$(B_w^* \mathcal{P})^* B_w^* \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + I \quad (49)$$

(and  $\lim_{s \rightarrow +\infty} B_w^* \mathcal{P}(s - A)^{-1}B = 0$ , see Lemma 6.5) and  $K = -B_w^* \mathcal{P}$  (and  $S = I$ ); thus, there is a nonnegative solution  $\mathcal{P} \in \mathcal{B}(H)$  to this problem iff the state-FCC is satisfied.

Note that the results mentioned in Corollary 8.3(b) [the above text in brackets corresponds to those results; such text must all be included or all excluded] correspond to two IREs (or AREs) each; e.g., Corollary 5.7(i) to (49) and to the *dual state-FCC ARE* (or filter ARE)  $(C_w \tilde{\mathcal{P}})^* C_w \tilde{\mathcal{P}} = A \tilde{\mathcal{P}} + \tilde{\mathcal{P}} A + I$ , whose unique nonnegative solution  $\tilde{\mathcal{P}} \in \mathcal{B}(H)$  provides  $H = \tilde{K}^* = (-I^{-1} C_w \tilde{\mathcal{P}})^* = -(C_w \tilde{\mathcal{P}})^*$ . Replace  $B$  by  $B_1$  for Corollary 5.3 or Remark 5.8 (in the latter, use also the dual with  $C_2$  in place of  $C$ ).

In Theorem 5.9(i) the ARE becomes the *output-FCC ARE*

$$K^* S K = A^* \mathcal{P} + \mathcal{P} A + C^* C \quad (50)$$

with  $K = -B_w^* \mathcal{P}$ ,  $S = D^* D$  (this leads to (iii) with  $\mathcal{N}^* \mathcal{N} + \mathcal{M}^* \mathcal{M} = S$ ; use  $[\frac{A}{S^{1/2}K} \mid \frac{B}{I-S^{1/2}}]$  to generate the  $[\frac{\mathcal{A}}{\mathcal{K}} \mid \frac{\mathcal{B}}{\mathcal{F}}]$  satisfying (iii) completely (cf. (28))). Thus,  $\hat{\mathcal{M}}(s) = I + K_w(s - A_\circ)^{-1}B$ ,  $\hat{\mathcal{N}}(s) = D + (C_\circ)_w(s - A_\circ)^{-1}B$ ,  $A_\circ = A + BK_w$ ,  $(C_\circ)_w = C_w + DK_w$  (use  $S^{-1/2} \mathcal{N}$ ,  $S^{-1/2} \mathcal{M}$  for (iii)), by Proposition 6.6.17(d4) of [M02].

Note that  $\hat{\mathcal{M}}(s) = I + K_w(s - A_\circ)^{-1}B$ ,  $\hat{\mathcal{N}}(s) = D + C_\circ(s - A_\circ)^{-1}B$ ,  $A_\circ = A + BK_w$ ,  $C_\circ = C_w + DK_w$ . Naturally, in Theorem 5.17 also the dual of (50) is used.

Above we gave the AREs corresponding to (b) (not maximally unbounded  $B$ ); to obtain the corresponding general AREs ( $B^*$  [or  $C^*$ ] possibly maximally unbounded), we have to add the w-lim terms to  $S$  [and  $\tilde{S}$ ].

Naturally, the results of (b) and (c) apply also to general WPLSs, if we use the resolvent AREs (see [M03b]) (or reciprocal AREs if  $\rho(A) \cap i\mathbb{R} \neq \emptyset$ ) instead of the ordinary ones. Those AREs have bounded coefficients (e.g.,  $(s - A)^{-1}$  in place of  $A$ ) and are equivalent to corresponding  $\mathcal{S}^t$ -IREs; in particular, any of these equations give constructive formulas for the feedback and factors.

**Proof of Corollary 7.5:** (a) In each of the results (or proofs), [two] some kind of “FCC condition[s]” is shown to be equivalent to the existence of certain  $J$ -optimal (possibly for modified  $\Sigma$  and  $J$ , see the proofs) state-feedback pair[s]. By Theorem 7.2(i)&(v), this holds iff the corresponding IRE[s] (i.e., that corresponding to the possibly modified  $\Sigma$  and  $J$  in the proofs) has a  $\mathcal{U}_*$ -stabilizing solution. But a  $\mathcal{U}_*$ -stabilizing solution of the IRE is that of the  $\mathcal{S}^t$ -IRE, which in turn implies the FCC ( $\mathcal{K}_{\text{opt}} x_0 \in \mathcal{U}_*(x_0) \forall x_0$ ). Conversely, here the FCC is also sufficient, by Theorem 5.1.

(b) This follows from (a) and Lemma 6.5.

(c)&(d) These will be proved on p. 64.  $\square$

By (c) above, the  $J$ -optimal state-feedback pair over  $\mathcal{U}_{\text{out}}$  often corresponds to the smallest nonnegative solution of the IRE. Much more generally, the  $J$ -optimal state-feedback pair over  $\mathcal{U}_{\text{exp}}$  (or  $\mathcal{U}_{\text{str}}$ ) corresponds to the greatest solution of the IRE:

**Theorem 7.6 (Maximal solution  $\mathcal{P}_{\text{max}}$ )** *A strongly internally stabilizing solution of the IRE is unique and greater than any admissible solution having  $S \geq 0$ .*

(The proof is given on p. 65. *Strongly internally stabilizing* means that  $([\mathcal{K} \mid \mathcal{F}])$  is admissible and  $\mathcal{A}_{\text{out}}^t x_0 \rightarrow 0$  as  $t \rightarrow +\infty$ ; thus, any exponentially (or  $\mathcal{U}_{\text{str}}$ -)stabilizing solution is strongly internally stabilizing. Similar results hold for the  $\mathcal{S}^t$ -IRE.)

If, e.g.,  $J \geq 0$  then any nonnegative admissible solution has  $S \geq 0$ , by (46b), hence then an exponentially stabilizing solution is the greatest admissible nonnegative solution. In fact, it is then the greatest nonnegative solution of the IRE. (By discretization ([M02], Section 13.4), one obtains similar results on all solutions of the IRE (including the WR solutions of the ARE), regardless of admissibility, because in discrete-time all solutions have a bounded “ $K$ ” and are hence admissible.)

**Notes for Section 7:** For decades, the Riccati operator has been shown to satisfy numerous integral equations including the three appearing in the IRE; see [S98b] for the case of jointly stabilizable and detectable WPLSs and p. 481 of [M02] for a list of earlier ones. Our contribution in [M02] was 1. to pick these three and to label them as the IRE, 2. to prove the sufficiency (and define the  $\mathcal{U}_*$ -stabilizing solutions), 3. to generalize the necessity and sufficiency to arbitrary WPLSs and  $\mathcal{U}_*$ ’s, 4. to observe that the IRE is exactly the discrete-time ARE (for the discretized system  $\begin{bmatrix} \mathcal{A}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{D}^t \end{bmatrix}$ ) and to use the connection for several uniqueness-type results (the discrete-time ARE is technically significantly simpler than the continuous-time one due to bounded “generators”). This then allowed us to derive similar results on the ARE.

We presented the equivalence “(i) $\Leftrightarrow$ (v)” of Theorem 7.2 in Theorem 9.9.1 of [M02]. The other conditions in Theorem 7.2 seem to be new and so does Theorem 7.1 (although, with the  $\Sigma_{\text{opt}}$ -IRE in place of the  $\mathcal{S}^t$ -IRE, Theorem 7.1 is essentially Theorem 9.7.1 of [M02]; cf. the notes to Section 9). However, the literature on Riccati equations is so abundant, that probably some special cases of most of the equations have appeared before; e.g., while we were writing these notes, it was pointed out to us that recently in [MSW03] (equation (3.4)) and [WST01] (equation (38)) it was shown that a  $\mathcal{S}$ -IRE-resembling equation  $(0, R, 0$  on the left-hand-sides) holds iff the WPLS is “ $(R, \mathcal{P}, J)$ -energy preserving”.

Except for coprimeness, most of Corollary 7.5 has been known for Pritchard–Salamon systems (see Theorems 3.3 and 3.4 of [PS87]).

For the ARE (50) with bounded  $B, C$ , (“ $\tilde{S} \geq 0$ ” is redundant and) it was already known that an exponentially stabilizing solution is maximal (see the notes on p. 853 of [M02]). Theorem 7.6 generalizes this but its proof does not apply to Riccati inequalities unlike that of Theorem 9.8.13 of [M02]. Example 8.4.13 of [M02] is due to Ilya Spitkovsky.

## 8 Smooth WPLSs

In this section we shall study systems for which  $\mathcal{A}B$  and  $C_w \mathcal{A}B$  are in  $L_\omega^1$ , or (slightly) more generally, for which  $\mathcal{B}\tau : u \mapsto x$  and  $\mathcal{D} : u \mapsto y$  are convolutions with  $L_\omega^1$  functions (plus the feedthrough  $D$ ) for some  $\omega \in \mathbb{R}$  (“ $\mathcal{B}\tau, \mathcal{D} \in \text{MTIC}_\omega^{L^1}$ ”). This is typically the case if  $\mathcal{A}$  is smoothing (e.g., analytic).

For such systems, one more equivalent condition in most results of Section 5 is that the ARE has a nonnegative solution. Moreover, the solution determines desired factorizations and (optimal) stabilizing state-feedback operators. The resulting closed-loop maps are also of the same form (hence ULR, by Theorem 8.1(c)). For similar results under alternative assumptions, see Section 6.2 and Corollary 7.5.

If  $\mathcal{D}u = Du + f * u \forall u \in L^2$ , where  $D \in \mathcal{B}(U, Y)$  and  $f \in L^1(\mathbb{R}_+; \mathcal{B}(U, Y))$ , then we say that  $\mathcal{D} \in \text{MTIC}^{L^1}(U, Y)$ . When  $\mathcal{A} = \text{MTIC}^{L^1}$ ,  $\mathcal{A} = \text{TIC}$  or similar, we set  $\mathcal{A}_\infty := \cup_{\omega \in \mathbb{R}} \mathcal{A}_\omega$ , where  $\mathcal{A}_\omega := \{e^{i\omega \cdot} \mathcal{D} e^{-i\omega \cdot} \mid \mathcal{D} \in \mathcal{A}\}$ , so that  $\mathcal{A}_{\omega'} \subset \mathcal{A}_\omega \subset \text{TIC}_\omega \forall \omega \in \mathbb{R} \cup$

$\{\infty\}$ ,  $\omega' \leq \omega$ . Thus,  $\mathcal{D} \in \text{MTIC}_\infty^{\text{L}^1}(U, Y)$  if  $\mathcal{D} = D + f*$ , where  $e^{-\omega \cdot} f \in \text{L}^1(\mathbb{R}_+; \mathcal{B}(U, Y))$  for some  $\omega \in \mathbb{R}$ . Naturally,  $\mathcal{E} \in \mathcal{A}$  means that  $\mathcal{E} \in \mathcal{A}(U, Y)$  for some Hilbert spaces  $U, Y$ , and  $\mathcal{A}(U)$  stands for  $\mathcal{A}(U, U)$ .

We start by noting that  $\text{MTIC}_\infty^{\text{L}^1}$  smoothness is inherited by the optimal closed-loop system, hence the IRE becomes equivalent to the ARE:

**Theorem 8.1** *Let  $\mathcal{A} = \text{MTIC}_\infty^{\text{L}^1}$ . Assume that  $\mathcal{B}\tau, \mathcal{D} \in \mathcal{A}_\infty$ .*

(a) *If there is a  $J$ -optimal state-feedback pair  $[\mathcal{K} \mid \mathcal{F}]$ , and  $S \in \mathcal{GB}(U)$ , then the following hold:*

(a1) *We have  $\mathcal{F}, \mathcal{F}_\circ, \mathcal{X}, \mathcal{M}, \mathcal{N}, \mathcal{D}_\circ, \mathcal{B}_\circ\tau \in \mathcal{A}_\infty$ , and  $S = D^*JD$ .*

(a2) *If  $\mathcal{C}^d\tau \in \mathcal{A}_\infty$ , then  $\mathcal{K}^d\tau, \mathcal{C}_\circ^d\tau, \mathcal{K}_\circ^d\tau \in \mathcal{A}_\infty$ , and  $\mathcal{C}_\circ^d\tau, \mathcal{K}_\circ^d\tau, \mathcal{B}_\circ\tau, \mathcal{D}_\circ, \mathcal{F}_\circ, \mathcal{M}, \mathcal{N} \in \mathcal{A}_\omega$  for any  $\omega > \omega_A$ . If  $\mathcal{U}_* \subset \mathcal{U}_{\text{exp}}$ , then  $\omega_A < 0$ .*

(b) *Assume that  $\mathcal{S}_{\text{PT}} \in \mathcal{GB}$ . Then the following condition is equivalent to conditions (i)–(vi) of Theorem 7.2:*

(vii) *The ARE (38) has a  $\mathcal{U}_*$ -stabilizing solution.*

*Moreover, if (vii) holds, then the w-lim in the ARE converges in norm to zero, hence then  $S = D^*JD \in \mathcal{GB}(U)$ .*

(c) *Any map in  $\mathcal{A}_\infty$  is ULR.*

(The proof is given on p. 61.)

In (a),  $S$  is the signature operator of the problem (e.g., the one appearing in any of (ii)–(vi) of Theorem 7.2); recall from Theorem 7.2(b) that  $\mathcal{S}_{\text{PT}} \in \mathcal{GB} \Rightarrow S \in \mathcal{GB}$ .

Note that always  $\mathcal{B}\tau \in \text{TIC}_\infty(U, H)$ . We have  $\mathcal{B}\tau \in \text{MTIC}_\infty^{\text{L}^1}(U, H)$  iff  $\pi_{[0,1)} \mathcal{A}B \in \text{L}^1([0, 1]; \mathcal{B}(U, H))$ , by (15) (and Lemma 6.8.1(c) of [M02]). However,  $\mathcal{B}\tau, \mathcal{D} \in \text{MTIC}_\infty^{\text{L}^1}$  does not imply that  $C_w \mathcal{A}^t B$  is defined for any  $t \geq 0$ . Nevertheless,  $C_w \mathcal{A}^t B \in \text{L}_\omega^1(\mathbb{R}_+; \mathcal{B}(U, H))$  implies that  $\mathcal{D} \in \text{MTIC}_\omega^{\text{L}^1}$ .

If  $C_w \mathcal{A}, \mathcal{A}B, C_w \mathcal{A}B$  are all locally  $\text{L}^1$ , then the assumptions of the theorem and the corollary below are satisfied:

**Lemma 8.2** *If  $\omega > \omega_A$  and  $\mathcal{A} = \text{MTIC}_\infty^{\text{L}^1}$ , then the following are equivalent:*

(i)  $\mathcal{B}\tau, \mathcal{C}^d\tau, \mathcal{D} \in \mathcal{A}_\infty$ .

(ii)  $\mathcal{B}\tau, \mathcal{C}^d\tau, \mathcal{D} \in \mathcal{A}_\omega$ .

(iii)  $\mathcal{A}B, C_w \mathcal{A}, C_w \mathcal{A}B$  are integrable over  $[0, 1]$ .

(By  $\mathcal{B}\tau \in \mathcal{A}_\infty$  we mean that  $\mathcal{B}\tau \in \mathcal{A}_\infty(U, H)$  (not  $\mathcal{A}_\infty(U, H_{-1})$ ); similarly for  $\mathcal{C}^d\tau, \mathcal{D}$ , (ii), (iii) and (iii').)

**Proof:** We have “(iii) $\Rightarrow$ (ii)”, by Lemma 6.8.5(a) of [M02], “(ii) $\Rightarrow$ (i)” is trivial, and “(i) $\Rightarrow$ (iii)” is given in Lemma 6.8.3 (with a slight modification in the proof of (c)).  $\square$

As in Corollary 7.5(b)&(c), also this kind of systems are stabilizable iff the corresponding ARE has a nonnegative solution:

**Corollary 8.3** *Assume that  $\mathcal{B}\tau, \mathcal{D} \in \mathcal{A}_\infty := \text{MTIC}_\infty^{\text{L}^1}$ .*

(a) *In Theorems 5.1 and 5.9(iii) and Corollaries 5.2, 5.3 and 5.10, the pair  $[\mathcal{K} \mid \mathcal{F}]$  (if any exists) can be chosen so that  $\mathcal{F}, \mathcal{F}_\circ, \mathcal{X}, \mathcal{M}, \mathcal{N}, \mathcal{D}_\circ, \mathcal{B}_\circ\tau \in \mathcal{A}_\infty$ .*

*For this pair,  $\mathcal{C}^d\tau \in \mathcal{A}_\infty \Rightarrow \mathcal{K}^d\tau, \mathcal{C}_\circ^d\tau, \mathcal{K}_\circ^d\tau \in \mathcal{A}_\infty$ .*

(b) *If  $\mathcal{C}^d\tau \in \mathcal{A}_\infty$ , then  $[\mathcal{K} \mid \mathcal{F}]$  and  $[\frac{\mathcal{H}}{\mathcal{G}}]$  (if such exist) in Corollary 5.7(i), Theorem 5.17 and Remark 5.8 can be chosen so that  $\mathcal{F}, \mathcal{X}, \mathcal{M}, \mathcal{N}, \mathcal{D}, \mathcal{B}\tau, \mathcal{C}^d\tau, \mathcal{K}^d\tau, \mathcal{E}, \mathcal{G}, \mathcal{H}\tau, \mathcal{X}_1, \mathcal{Y}_1, \tilde{\mathcal{X}}, \tilde{\mathcal{Y}} \in \mathcal{A}_\infty$ , and the same holds with the subindex  $L$  or  $\tilde{L}$  added (not defined for  $\mathcal{X}, \mathcal{M}, \mathcal{N}, \mathcal{X}_1, \mathcal{Y}_1, \tilde{\mathcal{X}}, \tilde{\mathcal{Y}}$ ).*

(c) *In (a) and (b), one more equivalent condition in any of the results mentioned above is that the corresponding ARE(s) have  $\mathcal{U}_*$ -stabilizing solutions. Except for Theorem 5.1, another equivalent condition is that the corresponding ARE(s) have nonnegative solutions with the lim converging uniformly to zero (the last paragraph of Corollary 7.5 applies). Moreover, Corollary 6.6 applies (even if  $B$  is maximally unbounded).*

(d) *Assume that  $\mathcal{C}^d\tau \in \mathcal{A}_\infty$ . In Corollaries 5.2 and 5.3, the pair mentioned in (a) also satisfies  $\mathcal{B}_\circ\tau, \mathcal{D}_\circ, \mathcal{F}_\circ, \mathcal{N}, \mathcal{M}, \mathcal{C}_\circ^d\tau, \mathcal{K}_\circ^d\tau \in \mathcal{A}_\omega$  for some  $\omega < 0$ . In Corollary 5.7(i) and Remark 5.8, the subindexed maps mentioned in (b) belong to  $\mathcal{A}_\omega$  for some  $\omega < 0$ .*

(The proof is given on p. 66. Note that  $\mathcal{A}_\omega \subset \mathcal{A}$  for  $\omega < 0$ .)

We observe from (d), Lemma 8.2 and Corollary 5.2 that if  $C_w \mathcal{A} B, \mathcal{A} B, C_w \mathcal{A}$  are  $L^1$  over  $[0, 1]$  and the state-FCC is satisfied, then there is an exponentially stabilizing state-feedback operator  $K$  for which the closed-loop system has an  $L^1(\mathbb{R}; \mathcal{B}(U, Y))$  impulse response  $(C_w \mathcal{A}_\cup B)$ . Theorem 6.7(2.) shows that we can have  $K \in \mathcal{B}(H, U)$  etc.

The assumption  $\mathcal{B}\tau, \mathcal{D} \in \text{MTIC}_\infty^{L^1}$  is equivalent to  $\mathcal{B}\tau, \mathcal{D} \in \text{MTIC}_\infty$ , where  $\text{MTIC} \subset \text{TIC}$  is the bigger class allowing for delays too, as one can deduce from Section 6.8 of [M02] ( $\mathcal{B}\tau$  cannot contain delays, and if it is  $\text{MTIC}_\infty^{L^1}$ , then neither can  $\mathcal{D}$ ). What if  $f \in L^1_{\text{strong}}$  (i.e.,  $f u_0 \in L^1 \forall u_0$ ) instead of  $L^1$ , where  $f = \mathcal{A} B, C_w \mathcal{A} B$ ? We do not know (unless, e.g.,  $C$  is bounded; see Hypothesis 9.2.2 of [M02]), but  $f \in L^2_{\text{strong}}$  is sufficient (see also Theorem 6.7(3.)):

**Remark 8.4** ( $\mathcal{A} = H^2_{\text{strong}}$ ) *The class  $\mathcal{A} = \{\mathcal{D} \mid \hat{\mathcal{D}}(\cdot - \epsilon) \in H^2_{\text{strong}}(\mathbb{C}^+; \mathcal{B}) \text{ for some } \epsilon > 0\} =: \mathcal{A}_{H^2}$  will also do in Theorem 8.1(a1)  $\mathcal{E}(b) \mathcal{E}(c)$ , (and in corresponding parts of Corollary 8.3(a)  $\mathcal{E}(c)$ ), and  $\mathcal{A} = \mathcal{A}_{H^2} \cap \mathcal{A}_{H^2}^1 =: \mathcal{A}_2$  will do in the whole theorem and corollary. Since  $\mathcal{B}\tau, \mathcal{D} \in \mathcal{A}_{H^2}$  iff Theorem 6.7(3.) holds, for either of these two classes the ARE can be replaced by the  $B_w^*$ -ARE (under  $D^* J D \in \mathcal{GB}$ ), see p. 33.*

Another valid choice in the theorem and corollary is  $\mathcal{A} = \text{MTIC}^{L^1, \mathcal{BC}} := \{D + f* \in \text{MTIC}^{L^1} \mid f(t) \text{ is compact for all } t, \text{ and } \mathcal{D} \in \mathcal{B}\}$  (or with “finite-dimensional” in place of “compact”).

(The proof is given on p. 66.) Here  $F \in H^2_{\text{strong}}(\mathbb{C}^+; \mathcal{B}(U, Y))$  iff  $F : \mathbb{C}^+ \rightarrow \mathcal{B}(U, Y)$  is holomorphic and  $\|F\|_{H^2_{\text{strong}}} := \sup_{u_0 \in U, r > 0} \|F(r + i\cdot)u_0\|_{L^2(\mathbb{R}; Y)} < \infty$ .

**Notes for Section 8:** For more on  $\text{MTIC}^{L^1}$ ,  $\text{MTIC}$  and the other classes, see [M02], e.g., Sections 2.6, 6.8 and 9.2, which also provide further results on optimization and closed-loop smoothness (but do not cover those presented here) and notes. The results there cover also the case where  $\Sigma \in \text{SOS}$  and  $\mathcal{D} \in \mathcal{A}$  (no assumptions on  $\mathcal{B}\tau$ ). Note that  $\text{MTIC}^{L^1}$  is often called the Wiener class and  $\text{MTIC}$  the Callier–Desoer class. These classes seem to have been studied mainly in the Pritchard–Salamon setting or in less general settings.

## 9 Generalized IREs and the $\text{Dom}(A_{\text{opt}})$ -ARE

This far we have mainly presented the setting and the main results; most proofs and accompanying minor results still remain. In this section, we shall study equivalent conditions for the existence of a  $J$ -optimal control in WPLS form (see Theorems 4.7 and 4.6 for sufficient conditions). In particular, we shall (1.) prove Theorem 7.1, (2.) generalize the  $\text{Dom}(A + BK)$ -ARE theory of [FLT88], and (3.) provide further results and tools for subsequent sections.

The key to (1.) is the  $\Sigma_{\text{opt}}$ -IRE (Theorem 9.1), which we show to be equivalent to the  $\widehat{\Sigma_{\text{opt}}}$ -IRE,  $r$ -shifted  $\Sigma_{\text{opt}}$ -IRE (57)–(58),  $\mathcal{S}^t$ -IRE and  $\mathcal{S}$ -IRE. Each of these five equivalent (systems of) equations leads to further results on the  $J$ -optimal control, such as the results in the previous sections (e.g., the ARE and the IRE) or as the resolvent RE of [M03b].

In Theorem 9.9 we generalize the ARE theory of [FLT88] (by Flandoli, Lasiecka and Triggiani) by deriving (from the  $\widehat{\Sigma_{\text{opt}}}$ -IRE) the (infinitesimal) ARE on  $\text{Dom}(A_{\text{opt}})$ , i.e., on the domain of the closed-loop semigroup generator (assuming only the regularity of the original system, not that of the optimal control).

We start by showing that a control in WPLS form (see Definition 3.2) is optimal over  $\mathcal{U}_*$  iff it is  $\mathcal{U}_*$ -stable and satisfies the RCC and the  $\Sigma_{\text{opt}}$ -IRE (52)–(53):

**Theorem 9.1 ( $\Sigma_{\text{opt}}$ -IRE)** *Assume that  $\mathcal{K}_0$  is a control in WPLS form, and  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ .*

- (a) *Then  $\mathcal{K}_0 x_0$  is  $J$ -optimal and  $\mathcal{P} = \mathcal{C}_0^* J \mathcal{C}_0$  iff  $\mathcal{K}_0 x_0 \in \mathcal{U}_*(x_0)$  for all  $x_0 \in H$  and the following hold (for all  $t > 0$ ):*



$$\langle \mathcal{B}^t u + \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{A}_0^t x_0 \rangle \rightarrow 0, \text{ as } t \rightarrow +\infty \quad (\forall x_0 \in H, u \in \mathcal{U}_*(0)), \quad (51)$$

$$0 = (\mathcal{D}^t)^* J \mathcal{C}_0^t + (\mathcal{B}^t)^* \mathcal{P} \mathcal{A}_0^t \in \mathcal{B}(H, L^2([0, t]; U)), \quad (52)$$

$$\mathcal{P} = \mathcal{A}_0^{t*} \mathcal{P} \mathcal{A}_0^t + \mathcal{C}_0^{t*} J \mathcal{C}_0^t \in \mathcal{B}(H). \quad (53)$$

We can make the following enhancements in (a):

(a1) We may replace (53) above by

$$\mathcal{P} = \mathcal{A}_0^{t*} \mathcal{P} \mathcal{A}_0^t + \mathcal{C}_0^{t*} J \mathcal{C}_0^t \in \mathcal{B}(H). \quad (54)$$

(a2) Equations (54) and (57) are equivalent.

(b1) The RCC (51) is redundant if  $\mathcal{U}_* \subset \mathcal{U}_{\text{exp}}$  or  $\mathcal{U}_* \subset \mathcal{U}_{\text{str}}$ .

(b2) The limit in (51) exists whenever  $\mathcal{C}_0 x_0 \in L^2$  and (52)–(53) hold (but it need not be zero).

(c) If  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  (resp.  $\mathcal{U}_* = \mathcal{U}_{\text{str}}$ ), then  $\Sigma_0$  is  $J$ -optimal iff  $\Sigma_0$  is exponentially (resp. strongly) stable and (52)–(53) hold.

(d1) Equation (53) holds iff

$$-A_0^* \mathcal{P} = \mathcal{P} A + C_0^* J C \in \mathcal{B}(\text{Dom}(A), \text{Dom}(A_0)^*). \quad (55)$$

(d2) Assume (53) and let  $\omega \geq \max\{\omega_A, \omega_{A_0}\}$ . Then (52) holds iff

$$0 = \hat{\mathcal{D}}(s)^* J \hat{\mathcal{C}}_0(z) + B^*(s - A)^{-*} \mathcal{P}(\bar{s} + A_0)(z - A_0)^{-1} =: T(s, z) \quad (56)$$

for some (equivalently, all)  $s, z \in \mathbb{C}_\omega^+$ .

(e) If we would allow for any  $\mathcal{U}_*$  satisfying Definition 8.3.2 of [M02] (instead of Standing Hypothesis 4.1), then (a2), (b1), (b2), (d1) and (d2) still hold (and (54) would be equivalent to (53) under (52)).

(f)  $\mathcal{K}_0$  is  $J$ -optimal and  $\mathcal{P} = \mathcal{C}_0^* J \mathcal{C}_0$  iff (RCC) holds,  $\mathcal{K}_0 x_0 \in \mathcal{U}_*(x_0) \quad \forall x_0 \in H$ , and for some (hence all)  $r > 0$  we have

$$\langle x_0, \mathcal{P} x_1 \rangle_H = \langle \mathcal{C}_0 x_0, J \mathcal{C}_0 x_1 \rangle_{L_r^2} + 2r \langle \mathcal{A}_0 x_0, \mathcal{P} \mathcal{A}_0 x_1 \rangle_{L_r^2} \quad (\forall x_0, x_1 \in H), \quad (57)$$

$$\langle \mathcal{C}_0 x_0, J \mathcal{D} \eta \rangle_{L_r^2} = -2r \langle \mathcal{A}_0 x_0, \mathcal{P} \mathcal{B} \tau \eta \rangle_{L_r^2} \quad (\forall x_0 \in H, \eta \in \mathcal{U}_*(0)). \quad (58)$$

(Above we may replace  $\mathcal{U}_*(0)$  by  $L_r^2(\mathbb{R}_+; U)$  if  $r > \max\{0, \omega_A, \omega_{A_0}\}$  (and  $r \geq \vartheta$  unless we give up sufficiency).)

We call (52)–(53) the  $\Sigma_{\text{opt}}$ -IRE for  $\Sigma$  and  $J$ . We call (55)–(56) the  $\widehat{\Sigma}_{\text{opt}}$ -IRE for  $\Sigma$  and  $J$ . We call  $\mathcal{K}_0$  (or  $\mathcal{P}$ )  $\mathcal{U}_*$ -stabilizing if  $\mathcal{K}_0 x_0 \in \mathcal{U}_*(x_0) \quad \forall x_0 \in H$  and the RCC holds.

(The proof is given on p. 44.)

Note that  $\Sigma_{\text{opt}}$ -IRE is equivalent to  $\widehat{\Sigma}_{\text{opt}}$ -IRE. In the theorem we require the  $\Sigma_{\text{opt}}$ -IRE to hold for all  $t > 0$ , but if it holds for some  $t > 0$  and the RCC holds, then it holds for all  $t \geq 0$ , as one observes from Proposition 9.8.7 and Lemma 14.2.1 (and Theorem 13.4.4(f2) and Remark 13.4.6) of [M02].

By (b1), “ $\mathcal{U}_{\text{exp}}$ -stabilizing” means the same as “exponentially stabilizing”. Similarly, “ $\mathcal{U}_{\text{str}}$ -stabilizing” means that  $\Sigma_{\text{opt}}$  is strongly stable.

However, “ $\mathcal{U}_{\text{out}}$ -stabilizing” means that  $\Sigma_{\text{opt}}$  is output-stable ( $\mathcal{C}_{\text{opt}} x_0, \mathcal{K}_{\text{opt}} x_0 \in L^2 \quad \forall x_0 \in H$ ) and the RCC holds (see Example 6.4). Intuitively, this “extra” condition is because now we have more candidate controls to be ruled out ( $\mathcal{U}_{\text{exp}} \subsetneq \mathcal{U}_{\text{out}}$ ).

For stable problems, the RCC takes the simple form  $\langle \mathcal{A}^t x_0, \mathcal{P} \mathcal{A}^t x_0 \rangle \rightarrow 0$ , as shown in [M97] (and in Proposition 9.8.11 of [M02]).

In Theorem 9.9 we shall use (d1)–(d2) to develop a nonintegral form of the Riccati equation (the “ $\text{Dom}(A_{\text{opt}})$ -ARE”). In [M03b] we shall use (f) to create the resolvent RE theory; (f) is also needed for the proof of Lemma 10.7, which is used to obtain Theorem 7.2(iv) and hence all the results of Section 5!

Next we list several lemmas that are needed in the proofs of Theorem 9.1 and many other results.

Algebraic (infinitesimal) Lyapunov-type equations can be equivalently written in integral forms and vice versa, as described below:

**Lemma 9.2** Let  $\begin{bmatrix} \mathcal{A}_k \\ \mathcal{C}_k \end{bmatrix} \in \text{WPLS}(\{0\}, H, Y)$  ( $k = 1, 2$ ),  $P \in \mathcal{B}(H)$  and  $\tilde{J} \in \mathcal{B}(Y)$ . Then

$$\langle A_1 x_1, P x_2 \rangle_{H_1} + \langle x_1, P A_2 x_2 \rangle_{H_1} + \langle C_1 x_1, \tilde{J} C_2 x_2 \rangle_{Y_1} \geq 0 \quad (x_1 \in \text{Dom}(A_1), x_2 \in \text{Dom}(A_2)) \quad (59)$$

$$\iff (\mathcal{A}_1^t)^* P \mathcal{A}_2^t + (\mathcal{C}_1^t)^* \tilde{J} \mathcal{C}_2^t \geq P \quad (t \in [0, +\infty)). \quad (60)$$

Moreover, we can, equivalently, replace “ $(t \in [0, +\infty))$ ” by “ $(t \in (0, \epsilon))$ ”, for any  $\epsilon > 0$ , or require (59) only for  $x_k \in \cap_{n \in \mathbb{N}} \text{Dom}(A_k^n)$ . All this also holds with “ $=$ ” or “ $\leq$ ” in place of “ $\geq$ ”.

Equation (59) is equivalent to

$$A_1^* P + P A_2 + C_1^* \tilde{J} C_2 \geq 0 \quad (\text{in } \mathcal{B}(\text{Dom}(A_2), \text{Dom}(A_1)^*)), \quad (61)$$

where  $\text{Dom}(A_k)$  is equipped with the graph topology and  $\text{Dom}(A_k)^*$  is its dual w.r.t. the pivot space  $H$ . (This is the standard convention; see p. 464 and Lemma A.3.24 of [M02] or Lemma 2.4 for details.)

**Proof:** (Further details and references are given on p. 464 of [M02].)

1° “ $\Leftarrow$ ”: Let  $x_k \in \text{Dom}(A_k)$  ( $k = 1, 2$ ). Then

$$(\mathcal{A}_k x_k)' = A_k \mathcal{A}_k x_k = \mathcal{A}_k A_k x_k \in \mathcal{C}(\mathbb{R}_+; H_k) \quad (k = 1, 2), \quad (62)$$

in particular,  $\mathcal{A}_k x_k \in \mathcal{C}(\mathbb{R}_+; \text{Dom}(A_k))$ . Consequently,  $\mathcal{C}_k x_k = C_k \mathcal{A}_k x_k \in \mathcal{C}(\mathbb{R}_+; Y)$  ( $k = 1, 2$ ).

Since  $f := \langle x_1, g x_2 \rangle$ , where  $g := (\mathcal{A}_1^t)^* P \mathcal{A}_2^t + (\mathcal{C}_1^t)^* \tilde{J} \mathcal{C}_2^t - P$ , satisfies  $f(0) = 0$ ,  $f \geq 0$  and  $f \in \mathcal{C}^1(\mathbb{R}_+)$ , we have  $f'(0) \geq 0$ , which implies that (59) holds.

2° “ $\Rightarrow$ ”: Assume that (59) holds on  $\text{Dom}(A_1^\infty) \times \text{Dom}(A_2^\infty)$ . Let  $a_k \in \text{Dom}(A_k^\infty) := \cap_{n \in \mathbb{N}} \text{Dom}(A_k^n)$  and  $t \geq 0$ . Set  $x_k := \mathcal{A}_k^t a_k \in \text{Dom}(A_k^\infty)$ , so that  $\mathcal{C}_k a_k = C_k x_k$  and  $(\mathcal{A}_k a_k)'(t) = A_k x_k$  ( $k = 1, 2$ ), as in 1°. By substituting these into (59), we obtain

$$\begin{aligned} 0 &\leq \langle \mathcal{A}_1'(t) a_1, P \mathcal{A}_2(t) a_2 \rangle + \langle \mathcal{A}_1(t) a_1, P \mathcal{A}_2'(t) a_2 \rangle + \langle C_1 \mathcal{A}_1(t) a_1, \tilde{J} C_2 \mathcal{A}_2(t) a_2 \rangle \\ &= \frac{d}{dt} \left[ \langle \mathcal{A}_1(t) a_1, P \mathcal{A}_2(t) a_2 \rangle_H + \int_0^t \langle C_1 \mathcal{A}_1(t) a_1, \tilde{J} C_2 \mathcal{A}_2(t) a_2 \rangle_Y dt \right] \\ &= \frac{d}{dt} \left[ \langle a_1, \mathcal{A}_1(t)^* P \mathcal{A}_2(t) a_2 \rangle_H + \langle a_1, (\mathcal{C}_1^t)^* \tilde{J} \mathcal{C}_2^t a_2 \rangle_H \right]. \end{aligned}$$

Thus, the expression in brackets must be increasing, hence for any  $t > 0$ , we have

$$\begin{aligned} &\langle a_1, \mathcal{A}_1(t)^* P \mathcal{A}_2(t) a_2 \rangle + \langle a_1, \mathcal{C}_1^* \tilde{J} \pi_{[0,t]} \mathcal{C}_2 a_2 \rangle \\ &\geq \langle a_1, \mathcal{A}_1(0)^* P \mathcal{A}_2(0) a_2 \rangle + \langle a_1, \mathcal{C}_1^* \tilde{J} \pi_{[0,0]} \mathcal{C}_2 a_2 \rangle = \langle a_1, P a_2 \rangle - 0. \end{aligned}$$

The same holds for  $a_1, a_2 \in H \times H$  too, because  $\text{Dom}(A_k^\infty)$  is dense in  $H$ .

3° The “moreover” claim can be observed from the above proofs; the claim on “ $\leq$ ” follows by replacing  $P$  by  $-P$  and  $\tilde{J}$  by  $-\tilde{J}$ ; the claim on “ $=$ ” follows from “ $\leq$ ” and “ $\geq$ ”.  $\square$

When using the “dynamic programming principle”, we often need the following:

**Lemma 9.3** Let  $x_0 \in H$  and  $u \in L_{\text{loc}}^2(\mathbb{R}_+; U)$ . Then  $u \in \mathcal{U}_*(x_0)$  iff  $\pi_+ \tau^t u \in \mathcal{U}_*(\mathcal{A}^t x_0 + \mathcal{B}^t u)$  for some (equivalently, all)  $t \geq 0$ .

This says that  $u$  is admissible for some initial state  $x(0) = x_0$  iff at some (hence any) moment  $t$  the remaining part of  $u$  is admissible for the current state  $x(t)$ . The proof of this lemma is where we explicitly use the hypothesis that  $\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix}$  is a WPLS.

**Proof:** Given  $t \geq 0$ , set  $u' := \pi_{[0,t]} u$ ,  $u'' := \pi_+ \tau^t u$ , so that  $u = u' + \tau^{-t} \pi_+ u''$  and  $x_t := \mathcal{A}^t x_0 + \mathcal{B}^t u = \mathcal{A}^t x_0 + \mathcal{B}^t u'$ . Obviously,  $u \in L_{\mathcal{B}}^2 \Leftrightarrow u'' \in L_{\mathcal{B}}^2$ . We have (recall that  $\tau^t u'' = \pi_- \tau^t u$ )

$$(\mathcal{C} x_t) + \mathcal{D} u'' = (\pi_+ \tau^t \mathcal{C} x_0 + \pi_+ \mathcal{D} \tau^t u'') + \pi_+ \mathcal{D} \pi_+ \tau^t u = \pi_+ \tau^t (\mathcal{C} x_0 + \mathcal{D} u) \quad (63)$$

hence  $\mathcal{C} x_t + \mathcal{D} u'' \in L^2$  iff  $\mathcal{C} x_0 + \mathcal{D} u \in L^2$ .

Analogously, we can show that  $\mathcal{Q}x_t + \mathcal{R}u'' \in Z^s$  iff  $\pi_+ \tau^t(\mathcal{Q}x_0 + \mathcal{R}u) \in Z^s$ , i.e., iff  $\mathcal{Q}x_0 + \mathcal{R}u \in Z^s$  (by Standing Hypothesis 4.1). Thus, we have shown that  $u \in \mathcal{U}_*(x_0) \Leftrightarrow u'' \in \mathcal{U}_*(x_t)$ . Since  $t \geq 0$  was arbitrary, this establishes the claim.  $\square$

For any “test function”  $\tilde{\eta} \in L^2([0, t]; U)$ , there is  $\eta \in \mathcal{U}_*(0)$  s.t.  $\pi_{[0, t)}\eta = \tilde{\eta}$  and the rest of  $\eta$  is optimal (if  $\mathcal{K}_0$  is):

**Lemma 9.4** *Assume that  $\mathcal{K}_0$  is a control in WPLS form s.t.  $\mathcal{K}_0 x_0 \in \mathcal{U}_*(x_0)$  for all  $x_0 \in H$ .*

*Then, for any  $t \geq 0$  and  $\tilde{\eta} \in L^2([0, t]; U)$ , we have  $P^t \tilde{\eta} := \eta := \pi_{[0, t)}\tilde{\eta} + \tau^{-t} \mathcal{K}_0 \mathcal{B}^t \tilde{\eta} \in \mathcal{U}_*(0)$ ,  $\pi_{[0, t)}\eta = \tilde{\eta}$  and  $\mathcal{D}\eta = \mathcal{D}^t \tilde{\eta} + \tau^{-t} \mathcal{C}_0 \mathcal{B}^t \tilde{\eta}$ .*  $\square$

(The claim “ $\in \mathcal{U}_*(0)$ ” follows from Lemma 9.3 by setting  $x_0 = 0$ , since  $\mathcal{K}_0 \mathcal{B}^t \tilde{\eta} \in \mathcal{U}_*(\mathcal{B}^t \tilde{\eta})$ . The claim on  $\mathcal{D}\eta$  is straight-forward.) The “dynamic programming principle” of this lemma will allow us to establish the Riccati equation through (68). The principle was based (through Lemma 9.3) on the requirement on  $Z^s$  in Hypothesis 4.1.

Note that  $P^t u$  keeps  $\pi_{[0, t)}u$  but replaces  $\pi_{[t, \infty)}u$  by the optimal input (if  $\mathcal{K}_0$  is optimal), hence  $(P^t)^2 = P^t$ , so that  $P^t$  is a projection  $L_{\text{loc}}^2(\mathbb{R}; U) \rightarrow \text{Ran}(P^t) \subset \mathcal{U}_*(0)$ .

We shall also need the following fact on how integral operator equation systems can be written in the frequency domain and vice versa:

**Lemma 9.5** ( $\mathcal{D}^{t*} J \mathcal{D}_0^t \Leftrightarrow \hat{\mathcal{D}}^* J \hat{\mathcal{D}}_0$ ) *Assume that  $\tilde{U}$  is a Hilbert space,  $\left[ \begin{smallmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & \mathcal{D}_0 \end{smallmatrix} \right]$  is a WPLS on  $(\tilde{U}, H, Y)$ , and  $\mathcal{P} \in \mathcal{B}(H)$ ,  $J \in \mathcal{B}(Y)$ . Let  $\alpha \geq \max\{\omega_A, \omega_{A_0}\}$ .*

(a) *We have (64) iff (65) holds.*

(b) *We have (64a) and (64c) iff (65a) and (65c) hold.*

Above we referred to the following equations:

$$\mathcal{C}^{t*} J \mathcal{C}_0^t = \mathcal{A}^{t*} \mathcal{P} \mathcal{A}_0^t - \mathcal{P} \quad \forall t \geq 0, \quad (64a)$$

$$\mathcal{D}^{t*} J \mathcal{D}_0^t = \mathcal{B}^{t*} \mathcal{P} \mathcal{B}_0^t \quad \forall t \geq 0, \quad (64b)$$

$$\mathcal{D}^{t*} J \mathcal{C}_0^t = \mathcal{B}^{t*} \mathcal{P} \mathcal{A}_0^t \quad \forall t \geq 0. \quad (64c)$$

$$C^* J C_0 = A^* \mathcal{P} + \mathcal{P} A_0, \quad (65a)$$

$$\hat{\mathcal{D}}(s)^* J \hat{\mathcal{D}}_0(z) = (z + \bar{s}) B^* (s - A)^{-*} \mathcal{P} (z - A_0)^{-1} B_0 \quad \forall s, z \in \mathbb{C}_\alpha^+ \quad (65b)$$

$$\hat{\mathcal{D}}(s)^* J C_0(z - A_0)^{-1} = B^* (s - A)^{-*} \mathcal{P} (\bar{s} + A_0)(z - A_0)^{-1} \quad \forall s, z \in \mathbb{C}_\alpha^+. \quad (65c)$$

(c) *We can have “for some  $s, z \in \mathbb{C}_\alpha^+$ ” in place of “ $\forall s, z \in \mathbb{C}_\alpha^+$ ” in (b). The same applies to (a) if  $J = J^*$ ,  $\mathcal{P} = \mathcal{P}^*$ ,  $A_0 = A$ ,  $C_0 = C$ ,  $B_0 = \pm B$  and  $\hat{\mathcal{D}}_0 = \pm \hat{\mathcal{D}}$ .*

(d) *In addition to (c),  $s, z \in \mathbb{C}_\alpha^+$  can be replaced by  $s \in \rho(A)$ ,  $z \in \rho(A_0)$  if we use characteristic functions in place of the transfer functions  $\hat{\mathcal{D}}, \hat{\mathcal{D}}_0$ .*

(e) *Drop the standing hypotheses on  $\Sigma$  for the moment. Assume, instead, that  $A : H \supset \text{Dom}(A) \rightarrow H$  and  $A_0 : H \supset \text{Dom}(A_0) \rightarrow H$  are linear operators on  $H$ ,  $s \in \rho(A)$ ,  $z \in \rho(A_0)$ ,  $B^* \in \mathcal{B}(\text{Dom}(A^*), U)$ ,  $C \in \mathcal{B}(\text{Dom}(A), Y)$ ,  $B_0^* \in \mathcal{B}(\text{Dom}(A_0^*), \tilde{U})$ ,  $C_0 \in \mathcal{B}(\text{Dom}(A_0), Y)$ ,  $\hat{\mathcal{D}}(s) \in \mathcal{B}(U, Y)$ ,  $\hat{\mathcal{D}}_0(z) \in \mathcal{B}(\tilde{U}, Y)$ . Extend  $\hat{\mathcal{D}}, \hat{\mathcal{D}}_0$  by setting  $\hat{\mathcal{D}}(\zeta) := \hat{\mathcal{D}}(s) + (\zeta - s)C(s - A)^{-1}(\zeta - A)^{-1}B \forall \zeta \in \rho(A)$ ,  $\hat{\mathcal{D}}_0(\zeta) := \hat{\mathcal{D}}_0(z) + (\zeta - z)C_0(z - A_0)^{-1}(\zeta - A_0)^{-1}B_0 \forall \zeta \in \rho(A_0)$ .*

*Then the equations in (65a) and (65c) hold for these  $s, z$  iff they hold for all  $s \in \rho(A)$ ,  $z \in \rho(A_0)$ . If  $J = J^*$ ,  $\mathcal{P} = \mathcal{P}^*$ ,  $A_0 = A$ ,  $C_0 = C$ ,  $B_0 = \pm B$  and  $\hat{\mathcal{D}}_0 = \pm \hat{\mathcal{D}}$ , then the equations in (65) hold for these  $s, z$  iff they hold for all  $s \in \rho(A)$ ,  $z \in \rho(A_0)$ .*

When applying (b), one may want to set  $\mathcal{B}_0 = 0$ ,  $\mathcal{D}_0 = 0$ . Note that the formulas in (e) also hold in (a)–(d) except that we should have  $\hat{\mathcal{D}}, \hat{\mathcal{D}}_0$  (the characteristic functions) in place of  $\hat{\mathcal{D}}, \hat{\mathcal{D}}_0$ .

**Proof:** (Actually (64b) and (65b) are equivalent, which can be shown as in the proof as in Proposition 9.11.3 of [M02].)

(b)  $1^\circ$  *Equations (64a) and (65a) are equivalent, by Lemma 9.2.*

2° “If”: Let  $\omega > \alpha$ ,  $x_0 \in H$  and  $u \in L^2_\omega(\mathbb{R}_+; U)$  be arbitrary. Set  $F(t) := \mathcal{B}^t u$ ,  $G(t) := \mathcal{P}\mathcal{A}_0^t x_0$ ,  $f(t) := \mathcal{D}^t u$ ,  $g(t) := J\mathcal{C}_0^t x_0$ . Write (65c) as

$$\hat{\mathcal{D}}(s)^* J\widehat{\mathcal{C}}_0(z) = \hat{\mathcal{B}}(s)^* \mathcal{P}(z + \bar{s} - (z - A_0))(z - A_0)^{-1} = (z + \bar{s})\hat{\mathcal{B}}(s)^* \mathcal{P}(z - A_0)^{-1} - \hat{\mathcal{B}}(s)^* \mathcal{P}I \quad (66)$$

to observe that Lemma B.4(i) is satisfied (apply (66) to  $\langle \hat{u}(s), \hat{\mathcal{D}}(s)^* J\widehat{\mathcal{C}}_0(z)x_0 \rangle_Y$ ), hence so is (v); set  $r = 0$  to obtain (64c) (since  $u, x_0$  were arbitrary). Combine this with 1° to obtain “if”.

3° “Only if”: Let  $\omega, u, x_0, F, G, f, g$  be as above, so that (65c) follows from Lemma B.4(i) (since  $\omega, u, x_0$  were arbitrary) once we establish (v).

3.1° *Case  $r < 0$* : Since  $\pi_{[0,t)}\tau^r\mathcal{D}\pi_+ = \pi_{[0,t)}\mathcal{D}\tau^r = \pi_{[0,t)}\mathcal{D}\pi_{[0,t)}\tau^r\pi_+$  for  $r < 0$ , we obtain from (64c), that  $\langle \pi_{[0,t)}\tau^r\mathcal{D}u, \pi_{[0,t)}J\mathcal{C}_0x_0 \rangle = \langle \tau^r u, \mathcal{D}^t J\mathcal{C}_0^t x_0 \rangle = \langle \tau^r u, \mathcal{B}^{t*}\mathcal{P}\mathcal{A}_0^t x_0 \rangle = \langle \mathcal{B}^{t+r}u, \mathcal{P}\mathcal{A}_0^t x_0 \rangle - 0 = \langle \mathcal{B}^{t+r}u, \mathcal{P}\mathcal{A}_0^t x_0 \rangle - \langle \mathcal{B}^r u, \mathcal{P}x_0 \rangle$ , i.e., (v) holds for  $r < 0$ .

3.2° Let  $r \geq 0$ . Now  $\pi_{[0,t)}\tau^r\mathcal{D}u = \pi_{[0,t)}\mathcal{D}\pi_+\tau^r u + \pi_{[0,t)}\mathcal{D}\pi_-\tau^r\pi_+u = \mathcal{D}^t\tau^r u + \pi_{[0,t)}\mathcal{C}\mathcal{B}^r\pi_+u$ , by 4. of Definition 2.1. Therefore, (64c) implies that hence  $\langle \pi_{[0,t)}\mathcal{D}\tau^r u, J\mathcal{C}_0^t x_0 \rangle = \langle \mathcal{D}^t\tau^r u + \mathcal{C}^t\mathcal{B}^r u, J\mathcal{C}_0^t x_0 \rangle = \langle \mathcal{B}^t\tau^r u + \mathcal{A}^t\mathcal{B}^r u, \mathcal{P}\mathcal{A}_0^t x_0 \rangle - \langle \mathcal{B}^r u, \mathcal{P}x_0 \rangle = \langle \mathcal{B}^{t+r}u, \mathcal{P}\mathcal{A}_0^t x_0 \rangle - \langle \mathcal{B}^r u, \mathcal{P}x_0 \rangle$ , by 1°, hence (v) holds for  $r \geq 0$  too (see (66)); thus, (65c) holds (for all  $s, z \in \mathbb{C}_\omega^+$ ; but also  $\omega > \alpha$  was arbitrary, hence for all  $s, z \in \mathbb{C}_\alpha^+$ ).

(a) The proof is analogous to 2°–3° of the proof of (b):

1° “If”: Assume (65). From (b) we obtain (64a) and (64c). Let  $\omega > \alpha$  and  $u, v \in L^2_\omega(\mathbb{R}_+; U)$  be arbitrary. Set  $F(t) := \mathcal{B}^t u$ ,  $G(t) := \mathcal{P}\mathcal{B}_0^t v$ ,  $f(t) := \mathcal{D}^t u$ ,  $g(t) := J\mathcal{D}_0^t v$ , so that (64b) follows from Lemma B.4(v) (since  $u, v$  were arbitrary), because (i) follows from (65b) (note that  $G(0) = 0$ ).

2° “Only if”: Assume (64). From (b) we obtain (65a) and (65c). With  $\omega, u, v, F, G, f, g$  as in 1°, we obtain (65b) as in 3° of the proof of (b).

(c)&(d)&(e) Observe first that, by (171) and the resolvent equation,

$$\hat{\mathcal{D}}_\Sigma(z) - \hat{\mathcal{D}}_\Sigma(s) = C[(z - A)^{-1} - (s - A)^{-1}]B. \quad (67)$$

1° (b): Assume (65a) and (65c) for some fixed  $s \in \mathbb{C}_\alpha^+$ , so that  $f(s) = g(s)$ , where  $f(s) := \hat{\mathcal{D}}(s)^* J\mathcal{C}_0$ ,  $g(s) := B^*(s - A)^{-*}\mathcal{P}(\bar{s} + A_0)$ . We have  $(z - A)^{-*}\bar{z} - (s - A)^{-*}\bar{s} = [(z - A)^{-*} - (s - A)^{-*}]A^*$ , hence  $g(z) - g(s) = B^*[(z - A)^{-*} - (s - A)^{-*}](A^*\mathcal{P} + \mathcal{P}A_0) \forall z$ . By (65a), this equals  $B^*[(z - A)^{-*} - (s - A)^{-*}]C^*JC$ , which equals  $f(z) - f(s)$ , by (67). Thus,  $f(z) = g(z)$  for all  $z \in \mathbb{C}_\alpha^+$ , hence (c) holds for (b) (including the claim on characteristic functions, just replace  $\mathbb{C}_\alpha^+$  by  $\rho(A)$  above).

2° (a): Assume that (65) holds for some fixed  $s_0$  in place of  $s$  and  $z$ . By 1°, equations (65a) and (65c) hold for any  $s, z \in \mathbb{C}_\alpha^+$ . But, by (67) and (65c), we get (here  $T_s := (s - A)^{-1}$ )  $\hat{\mathcal{D}}(s)^* J[\hat{\mathcal{D}}_0(z) - \hat{\mathcal{D}}_0(s)] = \hat{\mathcal{D}}(s)^* J\mathcal{C}_0[T_z - T_s]B_0 = B^*(z - A)^{-*}\mathcal{P}(\bar{s} + A_0)[T_z - T_s]B_0 = B^*(z - A)^{-*}\mathcal{P}[(\bar{s} + z)T_z - (\bar{s} + s)T_s]B_0$ . (We used the fact that  $T_z - T_s = (s - z)T_zT_s = 0$  maps  $\text{Ran}(B_0)$  to  $T_z[H] = \text{Dom}(A_0)$ .)

Thus, (65b) is equivalent under the change of  $z$  (i.e., for a fixed  $s$ , it holds for all  $z$  or for no  $z$ ). Take the adjoint of (65b) to observe that it is equivalent under the change of  $s$  too. Thus, if it holds for some pair  $s, z$ , then it holds for all  $s, z$ .  $\square$

**Proof of Theorem 9.1:** Trivially, condition  $\mathcal{K}_0 x_0 \in \mathcal{U}_*(x_0)$  ( $x_0 \in H$ ) is necessary. For the rest of the proof, we assume that this condition holds. Consequently,  $\mathcal{C}_0$  is stable and Theorem 8.3.9(a2) of [M02] holds.

1° “Only if”: Given  $\tilde{\eta} \in L^2([0, t]; U)$ , we have for  $\eta := \tilde{\eta} + \tau^{-t}\mathcal{K}_0\mathcal{B}^t\tilde{\eta}$  and any  $x_0 \in H$  that (note that  $\eta \in \mathcal{U}_*(0)$ , by Lemma 9.4)

$$0 = \langle \mathcal{C}x_0 + \mathcal{D}\mathcal{K}_0x_0, J\mathcal{D}\eta \rangle = \langle (\pi_{[0,t)} + \tau^{-t}\tau^t\pi_{[t,\infty)})\mathcal{C}_0x_0, J\mathcal{D}\eta \rangle \quad (68)$$

$$= \langle \pi_{[0,t)}\mathcal{C}_0x_0, J\mathcal{D}\eta \rangle + \langle \pi_+\tau^t\mathcal{C}_0x_0, J\mathcal{D}\tau^t\eta \rangle \quad (69)$$

$$= \langle \pi_{[0,t)}\mathcal{C}_0x_0, J\mathcal{D}\eta \rangle + \langle \mathcal{C}_0\mathcal{A}_0^t x_0, J\mathcal{C}_0\mathcal{B}^t\tilde{\eta} \rangle \quad (70)$$

$$= \langle \mathcal{C}_0^t x_0, J\mathcal{D}^t\tilde{\eta} \rangle + \langle \mathcal{A}_0^t x_0, \mathcal{P}\mathcal{B}^t\tilde{\eta} \rangle. \quad (71)$$

Thus, (52) holds.

By Definition 2.1(3.), we have  $\tau^t\mathcal{C}_0 = \pi_+\tau^t\mathcal{C}_0 + \pi_-\tau^t\mathcal{C}_0 = \mathcal{C}_0\mathcal{A}_0 + \tau^t\mathcal{C}_0^t$ , hence  $\mathcal{P} = \mathcal{C}_0^*\mathcal{J}\mathcal{C}_0 = \mathcal{A}_0^*\mathcal{C}_0^*\mathcal{J}\mathcal{C}_0\mathcal{A}_0 + (\mathcal{C}_0^t)^*\mathcal{J}\mathcal{C}_0^t$ , i.e., (54) holds; by (a1), it implies (53). Moreover,

since  $\mathcal{P} = \mathcal{C}_0^* J \mathcal{C}_0$ , we obtain from Definition 2.1(3.) that

$$\langle \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{A}_0^t x_0 \rangle = \langle \mathcal{C}_0 \mathcal{A}_0^t x_0, J \mathcal{C}_0 \mathcal{A}_0^t x_0 \rangle = \langle \pi_+ \tau^t \mathcal{C}_0 x_0, J \pi_+ \tau^t \mathcal{C}_0 x_0 \rangle \quad (72)$$

$= \langle \mathcal{C}_0 x_0, \pi_{[t, \infty)} J \mathcal{C}_0 x_0 \rangle \rightarrow 0$ , as  $t \rightarrow +\infty$ , which shows that the second term in (51) converges to zero.

Let now  $x_0 \in H$  and  $\eta \in \mathcal{U}_*(0)$  be arbitrary. Because  $\langle \pi_{[0, t)} \mathcal{D} \eta, J \mathcal{C}_0 x_0 \rangle \rightarrow \langle \mathcal{D} \eta, J \mathcal{C}_0 x_0 \rangle$ , as  $t \rightarrow \infty$ , equation (52) implies that

$$\langle \mathcal{B} \tau^t \eta, \mathcal{P} \mathcal{A}_0^t x_0 \rangle \rightarrow -\langle \mathcal{D} \eta, J \mathcal{C}_0 x_0 \rangle, \quad \text{as } t \rightarrow +\infty. \quad (73)$$

Because  $\eta \in \mathcal{U}_*(0)$  was arbitrary,  $J$ -optimality implies that (51) holds.

2° “If”: Assume that  $\mathcal{K}_0 x_0 \in \mathcal{U}_*(x_0) \forall x_0 \in H$  and that (51)–(54) hold.

The identity  $\mathcal{P} = \mathcal{C}_0^* J \mathcal{C}_0$  follows from (54) by letting  $t \rightarrow +\infty$  and using (51). From (73) and (51) we obtain that  $\langle \mathcal{D} \eta, J \mathcal{C}_0 x_0 \rangle = 0$ .

*Remark:* As in 1°, we actually obtain that  $\mathcal{A}_0^t \mathcal{P} \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{A}_0^t x_0 \rightarrow 0$ , as  $t \rightarrow \infty$ .

(a1) Equation (54) is equivalent to (53), because (53)\*–(54) =  $\mathcal{K}_0^*((\mathcal{B}^t)^* \mathcal{P} \mathcal{A}_0^t + (\mathcal{D}^t)^* J \mathcal{C}_0^t) = 0$  when (52) holds.

(a2) By Lemma 9.2, equation (54) is equivalent to (†)  $A_0^* \mathcal{P} + \mathcal{P} A_0 + C_0^* J C_0 = 0$  (in  $\mathcal{B}(\text{Dom}(A_0), \text{Dom}(A_0)^*)$ ), and (57) is equivalent to  $0 = \tilde{A}^* \mathcal{P} + \mathcal{P} \tilde{A} + \tilde{C}^* \tilde{J} \tilde{C} = A_0^* \mathcal{P} + \mathcal{P} A_0 - 2r \mathcal{P} + C_0^* J C_0 + I^*(2r \mathcal{P}) I = A_0^* \mathcal{P} + \mathcal{P} A_0 + C_0^* J C_0$  (set  $\tilde{J} := \begin{bmatrix} J & 0 \\ 0 & 2r \mathcal{P} \end{bmatrix}$ ,  $\tilde{C} := \begin{bmatrix} C_0 \\ I \end{bmatrix}$ ,  $\tilde{A} := A_0 - r$  and note that “ $\tilde{\mathcal{C}}^* \tilde{J} \tilde{\mathcal{C}} = \mathcal{C}_0^* e^{-2r \cdot} J \mathcal{C}_0 + \mathcal{A}_0^* e^{-2r \cdot} 2r \mathcal{P} \mathcal{A}_0$ ”).

(b1) Let  $x_0 \in H$  and  $\eta \in \mathcal{U}_*(0)$ . If  $\mathcal{U}_* \subset \mathcal{U}_{\text{str}}$ , then  $\mathcal{A}_0 x_0, \mathcal{B} \tau \eta \in \mathcal{C}_0(\mathbb{R}_+; H)$ , by Theorem 8.3.9(a2) of [M02], hence then (51) obviously holds.

If  $\mathcal{U}_* \subset \mathcal{U}_{\text{exp}}$ , then  $\mathcal{A}_0 x_0, \mathcal{B} \tau \eta \in L^2(\mathbb{R}_+; H)$ , hence then the limit in (73) cannot be nonzero in any case, so it must be zero (since it exists, by (73)).

(b2) Let  $t \rightarrow +\infty$  in (52)–(54) (see the proof of (a1) for (54)).

(c) “Only if” follows from  $\mathcal{K}_0 x_0 \in \mathcal{U}_*(x_0)$  and the Closed Graph Theorem (as shown in Theorem 8.3.9(a2) of [M02]) and “if” from (a)&(b1).

(d1) This follows from Lemma 9.2.

(d2) Apply ((d1) and) Lemma 9.5(b)&(c) with  $-J$  in place of  $J$ .

(e) Hypothesis 4.1 was only used to prove the two “ $J$ -optimal” equivalences, hence (e) holds.

(f) We shall use below the facts that  $\langle \mathcal{A}_0 x_0, \mathcal{P} \mathcal{B} \tau u \rangle_H$  is bounded for  $u \in \mathcal{U}_*$ , by the RCC, and that  $L^2(\mathbb{R}_+; Y) \subset L_r^2(\mathbb{R}_+; Y)$ .

By (a), (a1) and (a2), we only have to establish (the “hence all” and “replace” claims and) the equivalence between (52) and (58), and we may assume (57), (54) and the RCC, hence also that  $\mathcal{P} = \mathcal{C}_0^* J \mathcal{C}_0$  (use (54) and the RCC).

Let  $\eta \in L_{\text{loc}}^2(\mathbb{R}_+; U)$ ,  $t \geq 0$ ,  $x_0 \in H$  be arbitrary. By Lemma 9.4, we have  $u := \pi_{[0, t)} \eta + \tau^{-t} \mathcal{K}_0 \mathcal{B}^t \eta \in \mathcal{U}_*(0)$ , hence  $\langle \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{B}^t u \rangle$  is bounded, by the RCC, and  $\mathcal{D} u \in L^2 \subset L_r^2$ . We also note that  $\mathcal{A}_0 x_0, \mathcal{C}_0 x_0 \in L_r^2$ .

1° *A useful identity:* Since  $\langle f, \pi_{[t, \infty)} g \rangle_{L_r^2} = e^{-2rt} \langle \tau^t f, \pi_+ \tau^t g \rangle_{L_r^2}$ ,  $f, g \in L_r^2$ , we have (use Definition 2.1 and (57))

$$e^{2rt} \langle J \mathcal{C}_0 x_0, \pi_{[t, \infty)} \mathcal{D} u \rangle_{L_r^2} = \langle J \pi_+ \tau^t \mathcal{C}_0 x_0, \pi_+ \mathcal{D}(\pi_+ + \pi_-) \tau^t u \rangle_{L_r^2} \quad (74)$$

$$= \langle J \mathcal{C}_0 \mathcal{A}_0^t x_0, \pi_+ \mathcal{D} \mathcal{K}_0 \mathcal{B}^t u + \mathcal{C} \mathcal{B} \tau^t u \rangle_{L_r^2} \quad (75)$$

$$= \langle J \mathcal{C}_0 \mathcal{A}_0^t x_0, \mathcal{C}_0 \mathcal{B}^t u \rangle_{L_r^2} \quad (76)$$

$$= \langle \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{B}^t u \rangle_H - 2r \langle \mathcal{A}_0 \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{A}_0 \mathcal{B}^t u \rangle_{L_r^2} \quad (77)$$

$$= \langle \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{B}^t u \rangle_H - 2r e^{2rt} \langle \mathcal{A}_0 x_0, \pi_{[t, \infty)} \mathcal{P} \mathcal{B} \tau u \rangle_{L_r^2}, \quad (78)$$

because  $\mathcal{B} \tau(\pi_- + \pi_+) \tau^t u = \mathcal{A} \mathcal{B}^t u + \mathcal{B} \tau \mathcal{K}_0 \mathcal{B}^t u = \mathcal{A}_0 \mathcal{B}^t u$  and

$$\langle \mathcal{A}_0 x_0, \pi_{[t, \infty)} \mathcal{P} \mathcal{B} \tau u \rangle_{L_r^2} = \int_{s=t}^{\infty} e^{-2rs} \langle \mathcal{A}_0^s x_0, \mathcal{P} \mathcal{B} \tau^s u \rangle_H ds \quad (79)$$

$$= \int_{v=0}^{\infty} e^{-2rt} e^{-2rv} \langle \mathcal{A}_0^v \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{B} \tau^v \tau^t u \rangle_H dv \quad (80)$$

$$= e^{-2rt} \langle \mathcal{A}_0 \mathcal{A}_0^t x_0, \mathcal{P} \mathcal{B} \tau \tau^t u \rangle_{L_r^2}. \quad (81)$$

2° “If”: From (78) and (58) (with  $u$  in place of  $\eta$ ) we obtain that

$$\langle \mathcal{C}_0 x_0, J\pi_{[0,t)} \mathcal{D}u \rangle_{L_r^2} = \langle \mathcal{C}_0 x_0, J\mathcal{D}u \rangle_{L_r^2} - \langle \mathcal{C}_0 x_0, J\pi_{[t,\infty)} \mathcal{D}u \rangle_{L_r^2} \quad (82)$$

$$= -2r \langle \mathcal{A}_0 x_0, \mathcal{P}\mathcal{B}\tau u \rangle_{L_r^2} - e^{-2rt} \langle \mathcal{A}_0^t x_0, \mathcal{P}\mathcal{B}^t u \rangle_H + 2r \langle \mathcal{A}_0 x_0, \pi_{[t,\infty)} \mathcal{P}\mathcal{B}\tau u \rangle_{L_r^2} \quad (83)$$

$$= -2r \langle \mathcal{A}_0 x_0, \pi_{[0,t)} \mathcal{P}\mathcal{B}\tau u \rangle_{L_r^2} - e^{-2rt} \langle \mathcal{A}_0^t x_0, \mathcal{P}\mathcal{B}^t u \rangle_H. \quad (84)$$

Since two terms above are differentiable a.e., so must the third be too. Differentiate (84) w.r.t.  $t$  and multiply by  $e^{2rt}$  to obtain that a.e.

$$\langle \mathcal{C}_0 x_0, J\mathcal{D}u \rangle_Y(t) = -2r \langle \mathcal{A}_0^t x_0, \mathcal{P}\mathcal{B}^t u \rangle_H + 2r \langle \mathcal{A}_0^t x_0, \mathcal{P}\mathcal{B}^t u \rangle_H - \langle \mathcal{A}_0 x_0, \mathcal{P}\mathcal{B}\tau u \rangle_H'(t). \quad (85)$$

Integrate both sides ( $\int_0^t$ ) to obtain (52).

3° “Only if”: This follows by going 2° backwards.

4° “Hence all”: This follows from 3°, because (52) is independent of  $r > 0$ .

5° “Replace”: 5.1° “Only if”: this follows as above (our additional assumptions on  $r$  imply that  $\mathcal{D}u, \mathcal{B}\tau u, \mathcal{A}_0 x_0 \in L_r^2$ , so that, e.g., (82) is justified). (Note that here any  $r \geq 0$ , for which  $\mathcal{A}_0, \mathcal{B}, \mathcal{D}$  are  $r$ -stable, will do.) 5.2° “If”: With the additional assumption  $r \geq \vartheta$ , we obviously have  $\mathcal{U}_*(0) \subset L_r^2$ , hence sufficiency remains.  $\square$

Next we shall prove the equivalence of the  $\mathcal{S}^t$ -IRE and the  $\hat{\mathcal{S}}$ -IRE to the  $\Sigma_{\text{opt}}$ -IRE:

**Lemma 9.6 ( $\mathcal{S}^t$ -IRE &  $\hat{\mathcal{S}}$ -IRE)** *Make the assumptions of Theorem 9.1. Then the  $\Sigma_{\text{opt}}$ -IRE (52)–(53) holds iff  $\mathcal{P}, \mathcal{K}_0$  solve the  $\mathcal{S}^t$ -IRE (43). The  $\mathcal{S}^t$ -IRE holds for all  $t > 0$  iff the  $\hat{\mathcal{S}}$ -IRE holds for some  $s, z \in \mathbb{C}_\omega^+$  (equivalently, for all  $s, z \in \mathbb{C}_\omega^+$ ). We can above replace  $\mathbb{C}_\omega^+$  by  $\rho(A) \cap \rho(A_{\text{opt}})$  if replace  $\widehat{\mathcal{K}}_{\text{opt}}$  by  $K_{\text{opt}}(\cdot - A_{\text{opt}})^{-1}$  and  $\hat{\mathcal{D}}$  by  $\hat{\mathcal{D}}_\Sigma$ .*

By Theorem A.6, we have  $\overline{\mathbb{C}^+} \cap \rho(A) \subset \rho(A) \cap \rho(A_{\text{opt}})$  if  $\mathcal{K}_0$  is stable.

**Proof of Lemma 9.6:** (We write “0” in place of “opt” to shorten the formulas. Note from 1° that “ $\Sigma_{\text{opt}}$ -IRE  $\Leftrightarrow \mathcal{S}^t$ -IRE” actually holds for any single, fixed  $t > 0$  (but in 2° we only show that  $\mathcal{S}^t$ -IRE holds for all  $t > 0$  iff the  $\hat{\mathcal{S}}$ -IRE holds for all  $s, z \in \mathbb{C}_{\omega_A}^+$ .)

1°  $\Sigma_{\text{opt}}$ -IRE  $\Leftrightarrow \mathcal{S}^t$ -IRE: Equation (43c) is equivalent to (52), as one notices by substituting the identities  $\mathcal{C}_0^t = \mathcal{C}^t + \mathcal{D}^t \mathcal{K}_0^t$  and  $\mathcal{A}_0^t = \mathcal{A}^t + \mathcal{B}^t \mathcal{K}_0^t$  into (52). Similar substitution into (53) and use of (43c) shows that (53) is equivalent to (43a) (under (43c)).

2°  $\widehat{\Sigma_{\text{opt}}}$ -IRE  $\Leftrightarrow \hat{\mathcal{S}}$ -IRE: (Recall from Theorem 9.1(d1)&(d2) that  $\Sigma_{\text{opt}}$ -IRE is equivalent to  $\widehat{\Sigma_{\text{opt}}}$ -IRE holding for some  $s, z$ , equivalently, for all  $s, z$ , so it suffices to prove “ $\widehat{\Sigma_{\text{opt}}}$ -IRE  $\Rightarrow \hat{\mathcal{S}}$ -IRE” for arbitrary, fixed  $s, z \in \mathbb{C}_\omega^+$ .)

2.1° (56)  $\Leftrightarrow$  (44c) (under (44b)): Substitute  $\widehat{\mathcal{C}}_0 = \widehat{\mathcal{C}} + \hat{\mathcal{D}} \widehat{\mathcal{K}}_0$ ,  $(z - A_0)^{-1} = (z - A)^{-1} [I + B \widehat{\mathcal{K}}_0(z)]$  into (56) to obtain that (here  $\hat{\mathcal{B}}^* := B^*(s - A)^{-*}$ ,  $\hat{\mathcal{B}} := (z - A)^{-1} B$  etc.)

$$\hat{\mathcal{D}}^* J \widehat{\mathcal{C}} + \hat{\mathcal{D}}^* J \hat{\mathcal{D}} \widehat{\mathcal{K}}_0 = -\hat{\mathcal{B}}^* \mathcal{P}[(\bar{s} + z)(z - A_0)^{-1} - I] \quad (86)$$

$$= -\hat{\mathcal{B}}^* \mathcal{P}[(\bar{s} + z)((z - A)^{-1} + \hat{\mathcal{B}} \widehat{\mathcal{K}}_0) - I], \quad (87)$$

equivalently, (use (44b))

$$\hat{\mathcal{S}} \widehat{\mathcal{K}}_0 = -\hat{\mathcal{D}}^* J \widehat{\mathcal{C}} - \hat{\mathcal{B}}^* \mathcal{P}(s^* + z)(z - A)^{-1} - \hat{\mathcal{B}}^* \mathcal{P}, \quad (88)$$

which is a reformulation of (44c), because

$$(\bar{s} + z)(z - A)^{-1} - I = (\bar{s} + A)(z - A)^{-1}. \quad (89)$$

2.2°  $\widehat{\Sigma_{\text{opt}}}$ -IRE  $\Rightarrow \hat{\mathcal{S}}$ -IRE: By Lemma 9.2, equation (54) is equivalent to

$$0 = A_0^* \mathcal{P} + \mathcal{P} A_0 + C_0^* J C_0, \quad (90)$$

hence so is (55). Assume  $\widehat{\Sigma}_{\text{opt}}\text{-IRE}$ . Then (note that  $A_0(z - A_0)^{-1} = z(z - A_0)^{-1} - I$ )

$$0 = (s - A_0)^{-*}(A_0^*P + PA_0 + C_0^*JC_0)(z - A_0)^{-1} \quad (91)$$

$$= \widehat{\mathcal{C}}_0^* J \widehat{\mathcal{C}}_0 + (\bar{s} + z)(s - A_0)^{-*}P(z - A_0)^{-1} - P(z - A_0)^{-1} - (s - A_0)^{-*}P \quad (92)$$

$$= (\widehat{\mathcal{C}} + \widehat{\mathcal{D}}\widehat{\mathcal{K}}_0)^* J (\widehat{\mathcal{C}} + \widehat{\mathcal{D}}\widehat{\mathcal{K}}_0) + (\bar{s} + z)(s - A_0)^{-*}P(z - A_0)^{-1} \quad (93)$$

$$- P(z - A_0)^{-1} - (s - A_0)^{-*}P \quad (94)$$

$$= \widehat{\mathcal{C}}^* J \widehat{\mathcal{C}} + \widehat{\mathcal{K}}_0^* \widehat{\mathcal{S}} \widehat{\mathcal{K}}_0 + \widehat{\mathcal{K}}_0^* \left( \widehat{\mathcal{D}} J \widehat{\mathcal{C}} + (\bar{s} + z) \widehat{\mathcal{B}} P (z - A)^{-1} \right) + \left( \right)^* \widehat{\mathcal{K}}_0 \quad (95)$$

$$+ (\bar{s} + z)(s - A)^{-*}P(z - A)^{-1} - P(z - A)^{-1} - P \widehat{\mathcal{B}} \widehat{\mathcal{K}}_0 - (s - A)^{-*}P - \widehat{\mathcal{K}}_0^* \widehat{\mathcal{B}}^* P \quad (96)$$

$$= (s - A)^{-*}(C^*JC + A^*P + PA)(z - A)^{-1} + \widehat{\mathcal{K}}_0^* \widehat{\mathcal{S}} \widehat{\mathcal{K}}_0 - 2\widehat{\mathcal{K}}_0^* \widehat{\mathcal{S}} \widehat{\mathcal{K}}_0, \quad (97)$$

by (89) and (88), hence (44a) holds.

2.3°  $\widehat{\Sigma}_{\text{opt}}\text{-IRE} \Leftarrow \widehat{\mathcal{S}}\text{-IRE}$ : Use first 2.1° and then go 2.2° backwards.

3°  $\rho(A) \cap \rho(A_{\text{opt}})$ : The above proof still applies (see Theorem A.6), mutatis mutandis (note from the proof that in Theorem 9.1(d2) we could have “ $s \in \rho(A)$ ,  $z \in \rho(A_{\text{opt}})$ ” in place of “ $s, z \in \mathbb{C}_\omega^+$ ”).  $\square$

The following is straight-forward (cf. Lemma 3.8):

**Lemma 9.7** ( $\widehat{\mathcal{S}}\text{-IRE} \Leftrightarrow \widehat{\mathcal{S}}\text{-IRE}_\circ$ ) *Make the assumptions of Theorem 9.1. Let  $[\mathcal{K} \mid \mathcal{F}]$  be an admissible state-feedback pair for  $\Sigma$  with closed-loop system  $\Sigma_\circ$*

*Then  $\mathcal{K}_\circ = -\mathcal{K} + \mathcal{X}\mathcal{K}_0$  satisfies the  $\widehat{\mathcal{S}}\text{-IRE}$  for  $\left[\frac{\mathcal{A}_\circ}{\mathcal{C}_\circ} \mid \frac{\mathcal{B}_\circ}{\mathcal{D}_\circ}\right]$  and  $J$  iff  $\mathcal{K}_0$  satisfies the  $\widehat{\mathcal{S}}\text{-IRE}$  for  $\Sigma$  and  $J$ .*

The relation  $\mathcal{S}^t = \mathcal{S}_{\text{PT}}P^t$  connects  $\mathcal{S}^t$  to the uniqueness of optimal control:

**Lemma 9.8** ( $\mathcal{S}^t = \mathcal{S}_{\text{PT}}P^t$ ) *Let  $\mathcal{K}_{\text{opt}}$  be a  $J$ -optimal control in WPLS form. Define  $\mathcal{S}^t$  by the  $\mathcal{S}^t\text{-IRE}$ .*

- (a) *Then  $\mathcal{S}^t = \mathcal{S}_{\text{PT}}P^t = P^t\mathcal{S}_{\text{PT}} = P^t\mathcal{S}_{\text{PT}}P^t \forall t \geq 0$ .*
- (b) *The  $J$ -optimal control is unique some (hence all)  $x_0 \in H$  iff  $\mathcal{S}^t$  is one-to-one for some (hence all)  $t > 0$ .*
- (c) *If  $\mathcal{K}_{\text{opt}}$  is given by a state-feedback pair, then  $\mathcal{S}^t = \mathcal{X}^{t*}S\mathcal{X}^t$ , hence then  $\mathcal{S}^t$  is one-to-one iff  $S$  is.*

Whenever there is a  $J$ -optimal control for each  $x_0 \in H$ , we get similar results. In [M03b] we shall show that one more equivalent condition in (b) is that  $\widehat{\mathcal{S}}(s, s)$  is one-to-one for some (hence all)  $s \in \mathbb{C}_{\max\{0, \omega_A, \vartheta\}}^+$ .

**Proof:** (a) (This follows from Lemma 4.4(iv), but we give here a more direct proof.) Let  $\tilde{u}, v \in \mathcal{U}_*(0)$ . Set  $u := P^t\tilde{u}$ ,  $\eta := P^t\mathcal{B}^t v - \pi_+\tau^t v$ . Then  $\eta \in \mathcal{U}_*(\mathcal{B}^t v) - \mathcal{U}_*(\mathcal{B}^t v) = \mathcal{U}_*(0)$ , by Lemmata 9.3 and 4.2, and  $\pi_+\mathcal{D}\tau u = \mathcal{C}_{\text{opt}}\mathcal{B}^t u$ , by Lemma 9.4, hence

$$\langle \tau^{-t}\eta, \mathcal{S}_{\text{PT}}u \rangle = \langle J\mathcal{D}\eta, \mathcal{D}\tau^t u \rangle = \langle J\mathcal{D}\eta, \mathcal{C}_0\mathcal{B}^t u \rangle = 0, \quad (98)$$

by  $J$ -optimality. Thus,  $\langle v + \tau^{-t}\eta, \mathcal{S}_{\text{PT}}u \rangle = \langle v, \mathcal{S}_{\text{PT}}u \rangle$ . But  $v + \tau^{-t}\eta = P^t v$ . Since  $v, \tilde{u} \in \mathcal{U}_*(0)$  were arbitrary, we have  $(P^t)^*\mathcal{S}_{\text{PT}}P^t = \mathcal{S}_{\text{PT}}P^t$ , hence  $(P^t)^*\mathcal{S}_{\text{PT}}P^t = [(P^t)^*\mathcal{S}_{\text{PT}}P^t]^* = (P^t)^*\mathcal{S}_{\text{PT}}$ .

Finally,  $\mathcal{D}P^t u = \mathcal{D}^t u + \tau^{-t}\mathcal{C}_0\mathcal{B}^t u$  for all  $u \in L_{\text{loc}}^2$ , by Lemma 9.4, hence

$$\langle P^t v, \mathcal{S}_{\text{PT}}P^t u \rangle = \langle \mathcal{D}^t v, J\mathcal{D}^t u \rangle + \langle \mathcal{B}^t v, (\mathcal{C}_0^* J \mathcal{C}_0)\mathcal{B}^t u \rangle = \langle v, \mathcal{S}^t u \rangle. \quad (99)$$

(b) If  $u$  is  $J$ -optimal for  $x_0 = 0$  (i.e.,  $\mathcal{S}_{\text{PT}}u = 0$ ), then  $\mathcal{S}^t u = (P^t)^*\mathcal{S}_{\text{PT}}u \equiv 0 \forall t \geq 0$ . Conversely, if  $\mathcal{S}^t u = 0$  for some  $u \in L_{\text{loc}}^2$  and  $t > 0$ , then  $P^t u$  is  $J$ -optimal for  $x_0 = 0$  (since  $\mathcal{S}_{\text{PT}}P^t u = \mathcal{S}^t u = 0$ ). Thus,  $\mathcal{S}_{\text{PT}}$  is one-to-one iff  $\mathcal{S}^t$  is. Now (b) follows from Lemma 4.4(d)&(c).

(c) This holds because  $\mathcal{X}^t \mathcal{M}^t = I$  (see the IRE).  $\square$

If  $\mathcal{D}$  is regular and there is a unique optimal control (Theorem 4.7), then we can generalize the classical results (and those in [FLT88]) by showing that the ARE  $A^*\mathcal{P} + \mathcal{P}A + C^*C_w = \mathcal{P}BB_w^*\mathcal{P}$  is satisfied on  $\text{Dom}(A_{\text{opt}})$ , and that the optimal control is  $u(t) = -B_w^*\mathcal{P}x(t)$  a.e. (in the standard LQR problem where  $J = I$ ,  $D^*D = I$ ,  $D^*C = 0$ ), by (c)&(b) below:

**Theorem 9.9 (Dom( $A_{\text{opt}}$ )-ARE)** *Let  $\mathcal{K}_{\text{opt}}$  be a  $J$ -optimal control for  $\Sigma$  in WPLS form. Then*

$$-A_{\text{opt}}^*\mathcal{P} = \mathcal{P}A_{\text{opt}} + C_{\text{opt}}^*JC_{\text{opt}} \in \mathcal{B}(\text{Dom}(A_{\text{opt}}), \text{Dom}(A_{\text{opt}})^*), \quad (100)$$

$$-A_{\text{opt}}^*\mathcal{P} = \mathcal{P}A + C_{\text{opt}}^*JC \in \mathcal{B}(\text{Dom}(A), \text{Dom}(A_{\text{opt}})^*), \quad (101)$$

$$-A^*\mathcal{P} = \mathcal{P}A_{\text{opt}} + C^*JC_{\text{opt}} \in \mathcal{B}(\text{Dom}(A_{\text{opt}}), \text{Dom}(A)^*). \quad (102)$$

Recall that  $C_{\text{opt}} = C_c + D_cK_{\text{opt}}$  and  $A_{\text{opt}} = A + BK_{\text{opt}}$  on  $\text{Dom}(A_{\text{opt}})$ .

Assume, in addition, that  $\mathcal{D}$  is WR. Then

- (a) (“ $K_{\text{opt}} = -B^*\mathcal{P}$ ”)  $\mathcal{P} \in \mathcal{B}(\text{Dom}(A_{\text{opt}}), \text{Dom}(B_w^*))$ ,  $B_w^*\mathcal{P} = -D^*JC_{\text{opt}}$  on  $\text{Dom}(A_{\text{opt}})$ , and

$$(D^*JD)K_{\text{opt}} = -B_w^*\mathcal{P} - D^*JC_w \in \mathcal{B}(\text{Dom}(A_{\text{opt}}), U). \quad (103)$$

- (b) (“ $u_{\text{opt}} = -B^*\mathcal{P}x_{\text{opt}}$ ”)  $(K_{\text{opt}})_w x(t) = -(D^*JD)^{-1}(B_w^*\mathcal{P} + D^*JC_w)x(t) = (\mathcal{K}_{\text{opt}}x_0)(t)$  for a.e.  $t \geq 0$  and all  $x_0 \in H$ , if  $D^*JD \in \mathcal{GB}(U)$  (here  $x := x_{\text{opt}}(x_0) := \mathcal{A}_{\text{opt}}x_0$ ). In particular,  $\mathcal{P}x(t) \in \text{Dom}(B_w^*)$  a.e.

- (c) (ARE on  $\text{Dom}(A_{\text{opt}})$ ) If  $\mathcal{D}$  and  $\mathcal{D}^d$  are SR and  $D^*JD \in \mathcal{GB}(U)$ , then

$$A^*\mathcal{P} + \mathcal{P}A + C^*JC_w = (\mathcal{P}B + C^*JD)(D^*JD)^{-1}(D^*JC_w + B_w^*\mathcal{P}) \quad (104)$$

in  $\mathcal{B}(\text{Dom}(A_{\text{opt}}), \text{Dom}(A_{\text{opt}})^*)$ .

- (d) (ARE  $\Leftrightarrow J$ -optimal) Assume, instead, that  $\mathcal{K}_{\text{opt}}$  is a control in WPLS form,  $\mathcal{D} \in \text{WR}$ , and  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ .

Then  $\mathcal{K}_{\text{opt}}$  is  $J$ -optimal and  $\mathcal{P} = \mathcal{C}_{\text{opt}}^*J\mathcal{C}_{\text{opt}}$  iff (102) and (103) hold and  $K_{\text{opt}}$  is “ $\mathcal{U}_*$ -stabilizing” (i.e.,  $\mathcal{K}_{\text{opt}}x_0 \in \mathcal{U}_*(x_0)$  for all  $x_0 \in H$  and the RCC (51) holds).

(See Section 9.7 of [M02] for further details, results and notes.) Since  $\text{Dom}(A_{\text{opt}})$  is not known a priori, we are not satisfied by the above but go on to derive the IRE to finally arrive at the ARE presented in Section 6. However, both (infinitesimal) algebraic REs have their applications; for the above see, e.g., [LT00].

**Proof:** Apply Lemma 9.2 to (54), (53) and (53)\* to obtain (100), (101) and (102). The formulae for  $A_{\text{opt}}$  and  $C_{\text{opt}}$  are from Theorem A.6.

- (a) Multiply (56) by  $zx_0$ , where  $x_0 \in \text{Dom}(A_{\text{opt}})$ , and let  $z \rightarrow +\infty$  to obtain that

$$-\hat{\mathcal{D}}(s)^*JC_{\text{opt}}x_0 = B^*\bar{s}(s - A)^{-*}\mathcal{P}x_0 + B^*(s - A)^{-*}\mathcal{P}A_{\text{opt}}x_0 \quad (105)$$

Let  $\mathbb{R} \ni s \rightarrow +\infty$  to obtain that  $\mathcal{P}x_0 \in \text{Dom}(B_w^*)$  and  $-D^*JC_{\text{opt}}x_0 = B_w^*\mathcal{P}x_0 + 0$ . Since  $C_{\text{opt}} = C_w + DK_{\text{opt}}$ , we obtain (103).

(b)–(d) Theorem 9.7.3 of [M02] contains a slightly stronger form of this theorem. Therefore, we refer the long proofs of (b)–(d), and only remark that formally (b) and (c) follow from (103) and (100), and that (d) follows by going the backwards the above proofs.  $\square$

**Notes for Section 9:** Theorem 9.1(d2)&(f) and Lemmas 9.5 and 9.6 (in particular, the  $\widehat{\Sigma}_{\text{opt}}$ -IRE, the  $\mathcal{S}^t$ -IRE and the  $\mathcal{S}$ -IRE) seem to be new (see the notes to Section 7). We established most of the rest of this section in Sections 8.3 and 9.7 of [M02].

However, the necessity of (52)–(55) (and essentially Lemma 9.2) was already known for some cases; see, e.g., [S98b] for jointly stabilizable and detectable  $J$ -coercive (over  $\mathcal{U}_{\text{out}}$ ) WPLSs. Similarly, for the case of bounded  $C$  and the cost  $\|y\|_2^2 + \|u\|_2^2$ , most of Theorem 9.9 is contained in [FLT88] (with the additional (implicit) assumption that a suitable extension of  $B^*$  exists; we have shown here that assumption is redundant (using  $B_w^*$ )). See the notes on p. 465 of [M02] for further details.



## 10 IRE: details

In this section we shall prove Theorem 7.2 and further results on the IRE. We start by a generalization of the theorem (dropping the uniqueness requirement): the  $J$ -optimal state-feedback pairs are exactly the ones determined by the  $\mathcal{U}_*$ -stabilizing solutions of the IRE:

**Theorem 10.1 (IRE  $\Leftrightarrow J$ -optimal [  $\mathcal{K} \mid \mathcal{F}$  ])** *The following are equivalent:*

- (i) *There is a  $J$ -optimal state-feedback pair over  $\mathcal{U}_*$ .*
- (ii) *The IRE has a  $\mathcal{U}_*$ -stabilizing solution.*
- (iii) *The  $\widehat{\text{IRE}}$  has a  $\mathcal{U}_*$ -stabilizing solution.*

Moreover, the following hold:

- (a1) *Problems (ii) and (iii) have same solutions (and (b) if it has any solutions).*
- (a2) *A solution  $\mathcal{P}$  of (ii) is unique (and  $\mathcal{P} = \mathcal{C}_\circ^* J \mathcal{C}_\circ$ ), and corresponding pairs [  $\mathcal{K} \mid \mathcal{F}$  ] are exactly the  $J$ -optimal state-feedback pairs over  $\mathcal{U}_*$ .*
- (b) *There is a minimizing state-feedback pair over  $\mathcal{U}_*$  iff (ii) holds and  $\mathcal{J}(0, u) \geq 0$  for all  $u \in \mathcal{U}_*(0)$ .*
- (c) *Solutions of (ii) are exactly those  $\mathcal{U}_*$ -stabilizing solutions of the  $\Sigma_{\text{opt}}$ -IRE that correspond to a state-feedback pair (with  $\Sigma_0 = \Sigma_\circ [ \begin{smallmatrix} I \\ 0 \end{smallmatrix} ]$ ).*
- (d) *The operator  $S$  is one-to-one iff the  $J$ -optimal control is unique. If  $S$  is one-to-one, then all  $J$ -optimal pairs are given by (28).*
- (e) *If  $\mathcal{S}_{\text{PT}} \in \mathcal{GB}$ , then  $S \in \mathcal{GB}(U)$ ; moreover, if  $\mathcal{S}_{\text{PT}} \gg 0$ , then  $S \gg 0$ .*

(The proof is given on p. 52.)

As before,  $\mathcal{P}$  is the  $J$ -optimal cost operator (over  $\mathcal{U}_*$ ) and  $\mathcal{J}(x_0, u) := \langle y, Jy \rangle$ . In fact, with perturbation  $u_\circ$  to the closed-loop system (see Figure 3, p. 11), the cost becomes

$$\langle y, Jy \rangle_{L^2(\mathbb{R}_+; Y)} = \langle x_0, \mathcal{P}x_0 \rangle_H + \langle u_\circ, Su_\circ \rangle_{L^2(\mathbb{R}_+; U)}. \quad (106)$$

Here  $y := \mathcal{C}_\circ x_0 + \mathcal{D}_\circ u_\circ$  for any  $x_0 \in H$  and  $u_\circ \in L^2(\mathbb{R}_+; U)$  with compact support; if  $\mathcal{N} := \mathcal{D}_\circ$  is stable (e.g.,  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$ ), then  $S = \mathcal{N}^* J \mathcal{N}$ , and any  $u_\circ \in L^2(\mathbb{R}_+; U)$  will do above. (See Theorem 9.9.1 of [M02] for details and further results.)

We conclude that the  $J$ -optimal state-feedback pairs over  $\mathcal{U}_{\text{exp}}$  are exactly those exponentially stabilizing state-feedback pairs that satisfy the IRE (with  $\mathcal{P} := \mathcal{C}_\circ^* J \mathcal{C}_\circ$  and  $S := \mathcal{N}^* J \mathcal{N}$ ), equivalently, that satisfy the  $\Sigma_{\text{opt}}$ -IRE (with  $\Sigma_\circ [ \begin{smallmatrix} I \\ 0 \end{smallmatrix} ]$  in place of  $\Sigma_0$ ). By Example 8.4.13 of [M02], such pairs need not exist even if there is a unique  $J$ -optimal control for each initial state (since there the  $J$ -optimal control in WPLS form is not given by any (well-posed) state-feedback pair, despite  $J$ -coercivity). Thus, the  $\Sigma_{\text{opt}}$ -IRE is strictly more general than the IRE.

Note from Lemma 10.2 that in (iii) (and hence in Theorem 7.2(vi) too) it suffices to have a  $\mathcal{U}_*$ -stabilizing pair [  $\mathcal{K} \mid \mathcal{F}$  ] that satisfies (47) for some  $s = z \in \rho(A)$ .

The IRE is equivalent to the  $\widehat{\text{IRE}}$ :

**Lemma 10.2 ( $\widehat{\text{IRE}}$ )** *Let  $\Sigma_{\text{ext}} = [ \begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix} ]$  be a WPLS,  $\mathcal{P} \in \mathcal{B}(H)$ ,  $S \in \mathcal{B}(U)$ . Set  $\mathcal{X} := I - \mathcal{F}$ . Then the IRE (46) is satisfied iff the  $\widehat{\text{IRE}}$  holds for all  $s, z \in \mathbb{C}_{\omega_A}^+$ .*

Moreover, when  $\mathcal{P} = \mathcal{P}^*$  and  $S = S^*$ , the  $\widehat{\text{IRE}}$  (47) holds for all  $s, z \in \rho(A)$  iff it holds for some  $s, z \in \rho(A)$  (use  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}$  (resp.  $\hat{\mathcal{D}}_\Sigma$ ) in place of  $\hat{\mathcal{X}}$  (resp.  $\hat{\mathcal{D}}$ )).

Naturally, with a slight abuse of notation, by  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s)$  we refer to “ $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}$ ” :=  $\hat{\mathcal{X}}(\alpha) + (\alpha - s)(-K)(\alpha - A)^{-1}(s - A)^{-1}B = I - \mathcal{F}_{\Sigma_{\text{ext}}}(s)$ , the characteristic function of  $[ \begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{K} & \mathcal{F} \end{smallmatrix} ]$ . Recall from Lemma A.2 that the characteristic functions coincide with the transfer functions on  $\mathbb{C}_{\omega_A}^+$ .

We can write (47b) as  $\hat{\mathcal{X}}(s)^* S \hat{\mathcal{X}}(z) = \hat{\mathcal{D}}(s)^* J \hat{\mathcal{D}}(z) + (z + \bar{s}) \widehat{\mathcal{B}} \tau(s)^* \mathcal{P} \widehat{\mathcal{B}} \tau(z)$  on  $\mathbb{C}_{\omega_A}^+$  (the factor  $z + \bar{s}$  is due to the fact that  $\mathcal{B}^t \mathcal{P} \mathcal{B}$  refers to the adjoint (inner product) in  $H$ , not in  $L^2$ ; see the proof of Lemma B.4 for details). Similarly, (47c) equals  $\hat{\mathcal{X}}(s)^* S \hat{\mathcal{X}}(z) = -\hat{\mathcal{D}}(s)^* J \hat{\mathcal{C}}(z) - (z + s^*) \widehat{\mathcal{B}} \tau(s)^* \mathcal{P} \hat{\mathcal{A}}(z) + \widehat{\mathcal{B}} \tau(s)^* \mathcal{P}$ . In [M03b] we shall show how to prove

these equations for any optimal control in WPLS form (including the ill-posed ones) and how to interpret these as REs for a modified system with bounded generators.

In Theorem 6.2 we showed that the  $\widehat{\text{IRE}}$  is equivalent to the ARE if(f)  $\mathcal{D}$  and  $\mathcal{F}$  are WR.

**Proof of Lemma 10.2:** This follows from (a)&(c)&(d) of Lemma 9.5 through substitutions  $\mathcal{C} \mapsto \begin{bmatrix} \mathcal{C} \\ -\mathcal{K} \end{bmatrix}$ ,  $\mathcal{D} \mapsto \begin{bmatrix} \mathcal{D} \\ \mathcal{X} \end{bmatrix}$ ,  $\mathcal{A}_0 \mapsto \mathcal{A}$ ,  $\mathcal{B}_0 \mapsto -\mathcal{B}$ ,  $\mathcal{C}_0 \mapsto \begin{bmatrix} \mathcal{C} \\ -\mathcal{K} \end{bmatrix}$ ,  $\mathcal{D}_0 \mapsto \begin{bmatrix} -\mathcal{D} \\ -\mathcal{X} \end{bmatrix}$ ,  $J \mapsto \begin{bmatrix} -J & 0 \\ 0 & S \end{bmatrix}$ .  $\square$

We shall soon use the fact that a causal self-adjoint (hence static) “operator” “ $\mathcal{D}^*J\mathcal{D}$ ” is an element of  $\mathcal{B}$  even when  $\mathcal{D}$  is unstable, so that the “operator” is not well-defined on the whole  $L^2$  a priori:

**Lemma 10.3** ( $\mathcal{D}^*J\mathcal{D} = S$ ) *Let  $\mathcal{D} \in \text{TIC}_\infty(U, Y)$  and  $J = J^* \in \mathcal{B}(Y)$ . Assume that  $\mathcal{D}u \in L^2$  and  $\langle \mathcal{D}\pi_+v, J\mathcal{D}\pi_-u \rangle = 0$  for all  $u, v \in L^2_c$ . Then there is a unique  $S = S^* \in \mathcal{B}(U)$  s.t.  $\langle \mathcal{D}v, J\mathcal{D}u \rangle = \langle v, Su \rangle$  for all  $u, v \in L^2_c$ .*  $\square$

(This is Lemma 2.3.1 of [M02]; in the proof it was shown the operators  $S_t := (\mathcal{D}\pi_{[-t,t]})^*J\mathcal{D}\pi_{[-t,t]} \in \mathcal{B}(L^2([-t,t]; U))$  are restrictions of each other and can be extended to a static operator (“ $S$ ”). Note that for  $\mathcal{D} \in \text{TIC}$  the term  $\mathcal{D}^*J\mathcal{D}$  would be well defined and hence the lemma would be a well-known simple consequence of the Liouville Theorem.)

Next we list the connections between the IRE and its variants:

**Lemma 10.4** *Let  $S \in \mathcal{B}(U)$ , and  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ . Let  $[\mathcal{K} \mid \mathcal{F}]$  be an admissible state-feedback pair for  $\Sigma$ , and let  $\Sigma_\circ := \left[ \begin{array}{c|c} \mathcal{A}_\circ & \mathcal{B}_\circ \\ \hline \mathcal{C}_\circ & \mathcal{D}_\circ \end{array} \right] \in \text{WPLS}(U, H, Y \times U)$  be the corresponding closed-loop system. Set  $\mathcal{M} := (I - \mathcal{F})^{-1}$ ,  $\mathcal{N} := \mathcal{D}\mathcal{M} = \mathcal{D}_\circ$ .*

*We consider, for  $t \geq 0$ , the equations*

$$0 = \mathcal{D}^{t*}J\mathcal{C}_\circ^t + \mathcal{B}^{t*}\mathcal{P}\mathcal{A}_\circ^t, \quad (107)$$

$$0 = \mathcal{D}_\circ^{t*}J\mathcal{C}_\circ^t + \mathcal{B}_\circ^{t*}\mathcal{P}\mathcal{A}_\circ^t, \quad (108)$$

$$\mathcal{P} = \mathcal{A}_\circ^{t*}\mathcal{P}\mathcal{A}_\circ^t + \mathcal{C}_\circ^{t*}J\mathcal{C}_\circ^t, \quad (109)$$

$$\mathcal{P} = \mathcal{A}_\circ^{t*}\mathcal{P}\mathcal{A}_\circ^t + \mathcal{C}_\circ^{t*}J\mathcal{C}_\circ^t, \quad (110)$$

$$\pi_{[0,t]}S = \mathcal{N}^{t*}J\mathcal{N}^t + \mathcal{B}_\circ^{t*}\mathcal{P}\mathcal{B}_\circ^t, \quad (111)$$

$$S\mathcal{K}^t = -\left(\mathcal{N}^{t*}J\mathcal{C}_\circ^t + \mathcal{M}^{t*}\mathcal{B}^{t*}\mathcal{P}\mathcal{A}_\circ^t\right). \quad (112)$$

*Claims (a1)–(b3) hold:*

**(a1)** *For any  $t \geq 0$  we have (108) $\Leftrightarrow$ (107), as well as (46b) $\Leftrightarrow$ (111), and (46c) $\Leftrightarrow$ (112).*

**(a2)** *Any admissible solution of the IRE satisfies (107)–(112).*

**(b1)** *Let  $t \geq 0$  and let (112) hold. Then (110) $\Leftrightarrow$ (46a).*

**(b2)** *Let  $t \geq 0$  and let (46b) hold. Then (108) $\Leftrightarrow$ (112).*

**(b3)** *Let  $t \geq 0$  and let (108) hold. Then (109) $\Leftrightarrow$ (110).*

*If  $\mathcal{C}_\circ$  is stable, then (c1)–(c4) hold:*

**(c1)** *We have  $\mathcal{N}\pi_{[0,t]} \in \mathcal{B}(L^2)$  for all  $t \geq 0$ .*

**(c2)** *Assume that  $\mathcal{P} = \mathcal{C}_\circ^*J\mathcal{C}_\circ$ . Then (46b) is equivalent to*

$$\langle \mathcal{D}_\circ u, J\mathcal{D}_\circ v \rangle_{L^2(\mathbb{R}_+; U)} = \langle u, Sv \rangle_{L^2(\mathbb{R}_+; U)} \quad (u, v \in L^2([0, t]; U)). \quad (113)$$

*Moreover, (46b) holds for all  $t > 0$  iff*

$$\langle \hat{\mathcal{N}}u_0, J\hat{\mathcal{N}}u_0 \rangle_Y = \langle u_0, Su_0 \rangle \quad \text{a.e. on } i\mathbb{R} \quad (u_0 \in U). \quad (114)$$

**(c3)** *If  $\mathcal{P} = \mathcal{C}_\circ^*J\mathcal{C}_\circ$ , then (108) is equivalent to*

$$\langle \mathcal{D}_\circ\pi_+u, J\mathcal{C}_\circ x_0 \rangle_{L^2(\mathbb{R}_+; U)} = 0 \quad (u \in L^2([0, t]; U), x_0 \in H). \quad (115)$$

(c4) Assume that  $\mathcal{P} = \mathcal{C}_\circ^* J \mathcal{C}_\circ$  and that (108) holds for all  $t > 0$ .

Then there is a unique  $\tilde{S} \in \mathcal{B}(U)$  s.t.  $\langle \mathcal{N}u, J\mathcal{N}u \rangle = \langle u, \tilde{S}u \rangle$  ( $u \in L_c^2$ ).

Moreover,  $\tilde{S} = \tilde{S}^* \in \mathcal{B}(U)$ , and the IRE (46) and (107)–(115) are satisfied for all  $t \geq 0$  with  $\tilde{S}$  in place of  $S$ .

**Proof:** We first recall from (25) that

$$\mathcal{A}_\circ^t = \mathcal{A}^t + \mathcal{B}^t \mathcal{M}^t \mathcal{K}^t = \mathcal{A}^t + \mathcal{B}_\circ^t \mathcal{K}^t, \quad \mathcal{C}_\circ^t = \mathcal{C}^t + \mathcal{D}^t \mathcal{M}^t \mathcal{K}^t = \mathcal{C}^t + \mathcal{N}^t \mathcal{K}^t. \quad (116)$$

(a1) Multiply by  $\mathcal{M}^t$  or  $\mathcal{K}^t$  to the left.

(a2) Use (a1) and (b1).

(b1) Insert (112) into (46a) to obtain (110) (recall (116)).

(b2) From (116) and (111) (see (a1)) we obtain that

$$\mathcal{D}_\circ^{t*} J \mathcal{C}_\circ^t + \mathcal{B}_\circ^{t*} \mathcal{P} \mathcal{A}_\circ^t = S \mathcal{K}^t + \mathcal{N}^{t*} J \mathcal{C}^t + \mathcal{M}^{t*} \mathcal{B}^{t*} \mathcal{P} \mathcal{A}^t. \quad (117)$$

(b3) By (116), the difference (109)–(110)\* is equal to

$$\mathcal{K}^{t*} \left( \mathcal{D}_\circ^{t*} J \mathcal{C}_\circ^t + \mathcal{B}_\circ^{t*} \mathcal{P} \mathcal{A}_\circ^t \right) = \mathcal{K}^* 0 = 0. \quad (118)$$

(c1) Since  $\pi_+ \mathcal{N} \pi_- = \mathcal{C}_\circ \mathcal{B}_\circ$ , by Definition 2.14., we have

$$\pi_{[t,\infty)} \mathcal{N} \pi_{[0,t)} = \tau^{-t} \pi_+ \mathcal{N} \pi_- \tau^t \pi_{[0,t)} = \tau^{-t} \mathcal{C}_\circ \mathcal{B}_\circ \tau^t \pi_{[0,t)} \in \mathcal{B}(L^2). \quad (119)$$

Since  $\mathcal{N}^t \in \mathcal{B}(L^2)$ , we have  $\mathcal{N} \pi_{[0,t)} = \mathcal{N}^t + \pi_{[t,\infty)} \mathcal{N} \pi_{[0,t)} \in \mathcal{B}(L^2)$ .

(c2) (From (c1) it follows (see Lemma 2.1.13 of [M02] for more) that there is a holomorphic  $\hat{\mathcal{N}} : \mathbb{C}^+ \rightarrow \mathcal{B}(U, Y)$  s.t.  $\widehat{\mathcal{N}}u = \hat{\mathcal{N}}\hat{u}$  for all  $u \in L_c^2(\mathbb{R}_+; U)$  and that  $\hat{\mathcal{N}}u_0$  has a radial (even nontangential) limit a.e. for each  $u_0 \in U$  (indeed,  $\widehat{f}\hat{\mathcal{N}}u_0 = \widehat{\mathcal{N}f}u_0 \in H^2(\mathbb{C}^+; Y)$  when  $\widehat{f} \in L_c^2(\mathbb{R}_+)$ ). However, when  $\dim U = \infty$ , the map  $\hat{\mathcal{N}}$  need not have a boundary function (or it does, but the values are not in  $\mathcal{B}(U, Y)$  anywhere on  $i\mathbb{R}$ ). (e.g.,  $\hat{\mathcal{N}}$  could be the Cayley transform of  $F$  of Example 3.3.6 of [M02], multiplied by, e.g.,  $e^{-s^2/2}$ .)

1° Since  $\mathcal{P} = \mathcal{C}_\circ^* J \mathcal{C}_\circ$ , we obtain from (119) that

$$\langle \pi_{[t,\infty)} \mathcal{N} \pi_{[0,t)} u, J \pi_{[t,\infty)} \mathcal{N} \pi_{[0,t)} v \rangle = \langle \tau^{-t} \mathcal{C}_\circ \mathcal{B}_\circ \tau^t \pi_{[0,t)} u, J \tau^{-t} \mathcal{C}_\circ \mathcal{B}_\circ \tau^t \pi_{[0,t)} v \rangle \quad (120)$$

$$= \langle \mathcal{B}_\circ^t u, \mathcal{P} \mathcal{B}_\circ^t v \rangle \quad (u, v \in L^2). \quad (121)$$

Consequently,  $\langle u, \pi_{[0,t)} S v \rangle = \langle \mathcal{N} \pi_{[0,t)} u, (\pi_{[0,t)} + \pi_{[t,\infty)}) J \mathcal{N} \pi_{[0,t)} v \rangle$  for  $u \in L^2$  iff (111) holds (equivalently, (46b) holds, by (a1)).

2° Assume (113) (equivalently, (46b)) for all  $t > 0$ . Set  $u = f u_0, v = g u_0$ , where  $f, g$  are scalar to observe that  $\hat{f}^* \hat{g}$  (114) holds for all  $f, g \in L_c^2(\mathbb{R}_+)$ , hence (114) holds.

3° Assume (114). Obviously (Lemma A.3.1(g3) of [M02]), the latter  $u_0$ 's may be replaced by any  $v_0 \in U$ . If  $u = \chi_E u_0, v = \chi_F v_0$ , then (113) follows from the Plancherel Theorem. By linearity, we obtain (113) for simple functions, by density, for general  $u, v \in L_c^2$ , as required.

(c3) From the identity (use Definition 2.1)

$$\pi_{[0,t)} \tau^{-t} \mathcal{B}_\circ^* (\mathcal{C}_\circ^* J \mathcal{C}_\circ) \mathcal{A}_\circ(t) = \pi_{[0,t)} \tau^{-t} \pi_- \mathcal{D}_\circ^* \pi_+ J \pi_+ \tau^t \mathcal{C}_\circ \quad (122)$$

$$= \pi_{[0,t)} \mathcal{D}_\circ^* J \tau^{-t} \pi_+ \tau^t \mathcal{C}_\circ = \pi_{[0,t)} \mathcal{D}_\circ^* J \pi_{[t,\infty)} \mathcal{C}_\circ. \quad (123)$$

we obtain that the equation  $0 = \pi_{[0,t)} \mathcal{D}_\circ^* J \mathcal{C}_\circ = \pi_{[0,t)} \mathcal{D}_\circ^* J (\pi_{[0,t)} + \pi_{[t,\infty)}) \mathcal{C}_\circ$  is equivalent to (108), as claimed.

(c4) Now  $\langle \mathcal{N} \pi_+ v, J \mathcal{N} \pi_- u \rangle = \langle \mathcal{N} \pi_+ v, J \mathcal{C}_\circ \mathcal{B}_\circ u \rangle = 0$  for all  $u, v \in L_c^2$ , by (c3), hence there is a unique  $\tilde{S} = \tilde{S}^* \in \mathcal{B}(U)$  s.t.  $\langle \mathcal{N}u, J\mathcal{N}u \rangle = \langle u, \tilde{S}u \rangle$  ( $u \in L_c^2$ ), by (c1) and Lemma 10.3.

By (119), we have  $\mathcal{B}_\circ^{t*} \mathcal{P} \mathcal{B}_\circ^t = (\pi_{[t,\infty)} \mathcal{N} \pi_{[0,t)})^* J \pi_{[t,\infty)} \mathcal{N} \pi_{[0,t)}$ . It follows that (111) holds with  $\tilde{S}$  in place of  $S$ , for all  $t \geq 0$ .

As observed above (72), the identity  $\mathcal{P} = \mathcal{C}_\circ^* J \mathcal{C}_\circ$  leads to (109) for all  $t \geq 0$ ; by (c3), (108) holds for all  $t \geq 0$ . The remaining equations follow from (a1)–(b3).  $\square$

**Proof of Theorem 10.1:** 1° (i)⇒(ii): Assume (i), so that  $\Sigma_{\odot} \begin{bmatrix} I \\ 0 \end{bmatrix}$  solves the  $\Sigma_{\text{opt}}$ -IRE with  $\mathcal{P} := \mathcal{C}_{\odot}^* J \mathcal{C}_{\odot}$ , by Theorem 9.1, in particular, (107) holds. Now Lemma 10.4(a1)&(c4) provide us (108) and an  $S$  that completes  $\mathcal{P}, [\mathcal{K} \mid \mathcal{F}]$  to a solution of the IRE.

2° (ii)⇒(i): Obviously, a solution of (ii) is a solution of the  $\mathcal{S}^t$ -IRE, hence (i) follows from Lemma 9.6 and Theorem 9.1. (See Lemma 10.4 for an alternative proof.)

3° (ii)⇔(iii) and (a1): These follow from Lemma 10.2 (the term  $\mathcal{U}_*$ -stabilizing is defined for the  $\widehat{\text{IRE}}$  as for the IRE) and the proof of (b) (from which we see that if there are minimizing state-feedback pairs, then any  $J$ -optimal pairs are minimizing).

(a2) Uniqueness of  $\mathcal{P}$  follows from Lemma 4.4(a) and the rest from 1° and 2°.

(b) This follows from Lemma 4.4(d).

(c) This is obvious (see 1°).

(d) (Recall Lemma 4.4(c).) The “iff” holds because, by equation (9.175) (note: in (the last line of) Proposition 9.10.2(b3), one should the assumptions of (b4) (and apply (b1) in the proof)), we have  $\langle \mathcal{D}u, J \mathcal{D} \mathcal{X}^{-1} \eta \rangle = \langle \mathcal{X}u, S \eta \rangle$  for all  $\eta \in L_c^2(\mathbb{R}_+; U)$  and  $u \in \mathcal{U}_*(0)$  (hence  $S$  is one-to-one iff only  $u = 0$  is  $J$ -optimal for  $x_0 = 0$ ). Formula (28) follows from Lemma 3.7 (and we get  $E^{-*} S E^{-1}$  in place of  $S$ ).

(e) (For  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  also the converse holds, by Proposition 9.9.12 of [M02].) Fix  $t > 0$ . We have  $\mathcal{X}^t \in \mathcal{GB}(L^2([0, t]; U))$  (with inverse  $\mathcal{M}^t$ ) and  $\mathcal{S}^t = \mathcal{X}^{t*} S \mathcal{X}^t$ . If  $\mathcal{S}_{\text{PT}} \in \mathcal{GB}$ , then  $\mathcal{S}^t \in \mathcal{GB}$ , by Lemma 11.4(a), hence  $S \in \mathcal{GB}(U)$  (since  $\mathcal{X}^t \in \mathcal{GB}(L^2([0, t]; U))$  (with inverse  $\mathcal{M}^t$ ), and  $\mathcal{S}^t = \mathcal{X}^{t*} S \mathcal{X}^t$ ). Similarly,  $\mathcal{S}_{\text{PT}} \gg 0 \Rightarrow S \gg 0$  (cf. Lemma 12.2). (Note that the proof of Lemma 11.4 — actually, the whole Section 11 — is independent of this section).  $\square$

**Lemma 10.5** ( $\mathcal{S}^t$ -IRE &  $[\mathcal{K} \mid \mathcal{F}] \Rightarrow \widehat{\text{IRE}}$ ) *The admissible solutions of the IRE,  $\widehat{\text{IRE}}$ ,  $\mathcal{S}^t$ -IRE,  $\hat{\mathcal{S}}$ -IRE,  $\Sigma_{\text{opt}}$ -IRE and  $\widehat{\Sigma}_{\text{opt}}$ -IRE are the same.*

This means that if  $[\mathcal{K} \mid \mathcal{F}]$  is an admissible state-feedback pair for  $\Sigma$  and  $(\mathcal{P}, \mathcal{K}_{\odot})$  solves the  $\mathcal{S}^t$ -IRE (or the  $\Sigma_{\text{opt}}$ -IRE) for all  $t > 0$  (or the  $\hat{\mathcal{S}}$ -IRE or the  $\widehat{\Sigma}_{\text{opt}}$ -IRE for some  $s = z \in \mathbb{C}_{\omega}^+$ ), then there is  $S \in \mathcal{B}(U)$  s.t.  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$  is a solution of the IRE for all  $t > 0$  (and of the  $\widehat{\text{IRE}}$ ). Conversely, if  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$  is an admissible solution of the IRE (or of the  $\widehat{\text{IRE}}$ ), then  $(\mathcal{P}, \mathcal{K}_{\odot})$  solves the  $\mathcal{S}^t$ -IRE,  $\hat{\mathcal{S}}$ -IRE,  $\Sigma_{\text{opt}}$ -IRE and  $\widehat{\Sigma}_{\text{opt}}$ -IRE.

**Proof:** (By Lemma 9.6 and Theorem 9.1, the  $\mathcal{S}^t$ -IRE,  $\hat{\mathcal{S}}$ -IRE,  $\Sigma_{\text{opt}}$ -IRE and  $\widehat{\Sigma}_{\text{opt}}$ -IRE are equivalent. By Lemma 10.2, so are the IRE and the  $\widehat{\text{IRE}}$ .)

Since an admissible solution of the IRE is obviously one of the  $\mathcal{S}^t$ -IRE, it suffices to prove the converse. Let  $(\mathcal{P}, \mathcal{K}_{\odot})$  be an admissible solution of the  $\mathcal{S}^t$ -IRE.

Discretization of  $\mathcal{K}$  and  $\mathcal{X} := I - \mathcal{F}$  yields a solution of (14.10)–(14.12) of [M02] for  $[\begin{smallmatrix} \mathcal{A}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{D}^t \end{smallmatrix}]$  for a fixed  $t > 0$  (see p. 816 of [M02]), hence for  $nt$ ,  $n \in 1 + \mathbb{N}$ . By dediscretizing, from (14.11) we obtain that  $(\mathcal{X}^{nt})^* S^t \mathcal{X}^{nt} = \mathcal{S}^{nt}$ , i.e.,  $(\mathcal{X}^{-nt})^* \mathcal{S}^{nt} \mathcal{X}^{-nt} = S^t$  on  $[0, nt)$ , for any  $n \in \mathbb{N}$ , where  $S^t u := \sum_{k=0}^{\infty} \tau^{-k} S_t \tau^k \pi_{[0, t)} u$  and  $S_t \in \mathcal{GB}(L^2([0, t]; U))$  is the operator in (14.11). Obviously,  $\|S^t\|_{\mathcal{B}(L^2(\mathbb{R}_+; U))} = \|S\|$  and  $\tau^{-nt} S^t = S^t \tau^{-nt} \forall n \in \mathbb{N}$ .

Since the same holds with  $t/m$  in place of  $t$ , for any  $m \in 1 + \mathbb{N}$ , the corresponding we have  $\pi_{[0, t)} S^{t/m} = \pi_{[0, t)} (\mathcal{X}^{-mt/m})^* \mathcal{S}^{mt/m} \mathcal{X}^{-mt/m} = \pi_{[0, t)} S^t$ , hence  $S^t = S^{t/m}$ , hence  $\tau^{-nt/m} S^t = S^t \tau^{-nt/m} \forall n, m \in 1 + \mathbb{N}$ , hence  $\tau^{-T} S^t = S^t \tau^{-T} \forall T \geq 0$ , by continuity. By Lemma 2.1.3 of [M02],  $S^t$  has a unique extension to an element of  $\text{TIC}(U)$ . By continuity,  $(\mathcal{X}^T)^* S^t \mathcal{X}^T = \mathcal{S}^T \forall T > 0$ . Since  $S^t = (S^t)^*$  and  $\pi_{[t, \infty)} S^t \pi_{[0, t)} = 0$ , it follows from Lemma 2.3.2 of [M02] that  $S^t \in \mathcal{B}(U)$ ; thus,  $(\mathcal{P}, S^t, [\mathcal{K} \mid \mathcal{F}])$  solve the IRE.  $\square$

In Lemma 10.7 we shall prove the remaining part of Theorem 7.2. For the lemma, we need the following auxiliary result:

**Lemma 10.6 (Generalized SpF)** *Let  $\mathcal{K}_0$  be a control in WPLS form for  $\Sigma$  and  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ . Assume the  $\mathcal{S}^t$ -IRE (43) (or  $\Sigma_{\text{opt}}$ -IRE) for all  $t > 0$ . Then the following are equivalent (for this fixed  $\mathcal{P}$ ):*

- (i) *There is a solution of the IRE (46).*
- (ii) *Problem (45) has a solution  $\hat{\mathcal{X}} \in H_{\infty}^{\infty}(U)$ ,  $S = S^* \in \mathcal{B}(U)$  on some right half-plane.*

- (iii) There are  $\mathcal{X} \in \text{TIC}_\infty(U)$ ,  $S = S^* \in \mathcal{B}(U)$  satisfying  $\mathcal{S}^t = \mathcal{X}^{t*} S \mathcal{X}^t$  for all  $t > 0$ .

Moreover, the following hold:

- (a) The solutions (if any) of (i), (ii) and (iii) are the same (set  $\mathcal{K} := \mathcal{X} \mathcal{K}_0$ ,  $\mathcal{F} := I - \mathcal{X}$ , or, conversely,  $\mathcal{X} := I - \mathcal{F}$ ).
- (b) If (ii) holds,  $S$  is one-to-one and  $\hat{\mathcal{X}} \in \mathcal{GH}_\infty(U)$ , then  $[\mathcal{K} \mid \mathcal{F}]$  is an admissible state-feedback pair for  $\Sigma$  and  $\Sigma_0 = \Sigma_\diamond [I]$ .

**Proof of Lemma 10.6** (Actually, it would suffice to assume the  $\mathcal{S}^t$ -IRE and (iii) on any unbounded subset of  $[0, \infty)$ , since then it would still hold for all  $t \geq 0$ , as one observes from 3°–4° below. By Lemma 9.6, the  $\mathcal{S}^t$ -IRE and the  $\Sigma_{\text{opt}}$ -IRE are equivalent.)

1° (i)  $\Rightarrow$  (iii): This is trivial.

2° (iii)  $\Rightarrow$  (i): Set  $\mathcal{K} := \mathcal{X} \mathcal{K}_0$ ,  $\mathcal{F} := I - \mathcal{X}$  to obtain (i) from the  $\mathcal{S}^t$ -IRE (because  $\mathcal{S}^t \mathcal{K}_0^t = \mathcal{X}^{t*} S \mathcal{K}^t$ ,  $\mathcal{K}_0^{t*} \mathcal{S}^t \mathcal{K}_0^t = \mathcal{K}^{t*} S \mathcal{K}^t$ , by (46b)).

3° (iii)  $\Rightarrow$  (ii): (Note that this would follow from Lemma 10.2 (and (47b)) if we assumed that  $[\mathcal{K} \mid \mathcal{F}]$  extends  $\Sigma$  to another WPLS.) As in the proof of Lemma 11.4(b), we observe that (45) holds for all  $s \in \mathbb{C}_\omega^+$ , where  $\omega \geq \omega_A$  is s.t.  $\mathcal{X} \in \text{TIC}_\omega$ .

4° (ii)  $\Rightarrow$  (iii): Let  $\omega > \omega_A$  be s.t.  $\hat{\mathcal{X}} \in \mathcal{H}_\omega^\infty$ . Let  $u, v \in W_\omega^{1,2}(\mathbb{R}_+; U)$  (i.e.,  $u, u' \in L_\omega^2(\mathbb{R}_+; U)$  and  $u(t) = u(0) + \int_0^t u'(r) dr \forall t > 0$ ; similarly for  $v$ ).

Set  $g_1(t) := (\mathcal{B}^t v)'(t) = \mathcal{B}^t v' \in L_\omega^2$  (by Theorem 3.1.5 of [M02], since  $\mathcal{B} \tau \in \text{TIC}_\omega$ ). Then  $\hat{g}_1(s) = s(s - A)^{-1} B \hat{v}(s) - 0$ , by Lemma B.2 and Lemma A.2(d). Set  $f_1(t) := \mathcal{P} \mathcal{B} \tau^t u$ ,  $f_2(t) := (\mathcal{P} \mathcal{B} \tau^t u)'(t)$ ,  $g_2(t) := \mathcal{B} \tau^t v$ , so that

$$\langle \mathcal{B}^t u, \mathcal{P} \mathcal{B}^t v \rangle_H = \int_0^t \langle \mathcal{B}^t u, \mathcal{P} \mathcal{B}^t v \rangle'_H(t) dt = \int_0^t (\langle f_1(t), g_1(t) \rangle_H + \langle f_2(t), g_2(t) \rangle) dt. \quad (124)$$

Set  $F := \begin{bmatrix} -J \mathcal{D} u \\ S \mathcal{X} u \end{bmatrix}$ ,  $G := \begin{bmatrix} \mathcal{D} v \\ \mathcal{X} v \end{bmatrix}$  to obtain from Lemma B.3 (for which it suffices to have (45) on  $\mathbb{C}_\alpha \setminus \mathbb{C}_\beta$ ) that  $\langle F, G \rangle_{Y \times U} = \langle f, g \rangle_{H \times H}$  a.e. Take  $\int_0^t$  of both sides to obtain (46b) (since  $\pi_{[0,t]} W_\omega^{1,2}(\mathbb{R}_+; U) = W_\omega^{1,2}([0, t]; U)$  is dense in  $L^2([0, t]; U)$ , by Theorem B.3.11(b1) of [M02]).

(a) By 1°–4°, any solution of (i), (ii) or (iii) is a solution of all of them.

(b) 1° *Useful equations*: Since  $\pi_{[t,s]} \tau^T = \tau^T \pi_{[t+T, s+T]}$  for all  $t, s, T \in \mathbb{R}$ , we have for all  $T, t \geq 0$  that

$$\pi_{[0,t]} \tau^T ((\mathcal{X}^{T+t})^* S \mathcal{X}^{T+t}) \tau^{-T} \pi_{[-T,0]} = \pi_{[0,t]} \mathcal{X}^* S \tau^T \pi_{[0,T+t]} \tau^{-T} \mathcal{X} \pi_{[-T,0]} \quad (125)$$

$$= \mathcal{X}^{t*} S \pi_{[0,t]} \mathcal{X} \pi_{[-T,0]}, \quad (126)$$

because  $\tau^T \pi_{[0,T+t]} \tau^{-T} = \pi_{[-T,t]}$  and  $\pi_+ \mathcal{X}^* = \pi_+ \mathcal{X}^* \pi_+$ . Since  $\tau^{T+t} \pi_+ \tau^{-T} \pi_{[-T,0]} = \tau^t \pi_{[-T,0]}$  and  $\pi_{[0,t]} \tau^T \pi_+ \tau^{-t-T} = \pi_{[0,t]} \tau^{-t}$ , (125) equals (substitute  $t+T$  in place of  $t$  in (46b))

$$\mathcal{X}^{t*} S \pi_{[0,t]} \mathcal{X} \pi_{[-T,0]} = \mathcal{D}^{t*} J \pi_{[0,t]} \mathcal{D} \pi_{[-T,0]} + \mathcal{B}^{t*} \mathcal{P} \mathcal{B} \tau^t \pi_{[-T,0]}. \quad (127)$$

From (46c) we obtain that

$$\mathcal{X}^{t*} S \mathcal{K}^t \mathcal{B} = -\mathcal{D}^{t*} J \pi_{[0,t]} \mathcal{C} \mathcal{B} - \mathcal{B}^{t*} \mathcal{P} \mathcal{A}^t \mathcal{B} = -\mathcal{D}^{t*} J \pi_{[0,t]} \mathcal{D} \pi_- - \mathcal{B}^{t*} \mathcal{P} \mathcal{B} \tau^t \pi_- \quad (128)$$

(use 2.&4. of Definition 2.1). By (127), it follows that

$$-\mathcal{X}^{t*} S \mathcal{K} \mathcal{B} \pi_{[-T,0]} = \mathcal{X}^{t*} S \pi_+ \mathcal{X} \pi_{[-T,0]}. \quad (129)$$

2° We have  $-\mathcal{K} \mathcal{B} = \pi_+ \mathcal{X} \pi_-$  on  $L_c^2$ : This follows from (129) (given  $u \in L_c^2(\mathbb{R}; U)$ , choose  $T$  s.t.  $\pi_- u = \pi_{[-T,0]} u$ ), because  $\mathcal{X}^{t*} S$  is one-to-one (obviously,  $\pi_{[0,t]} \mathcal{X}^{-*} \pi_{[0,t]} = (\mathcal{X}^t)^{-*}$ ).

3° We have  $\pi_+ \tau^t \mathcal{K} = \mathcal{K} \mathcal{A}^t$  ( $t \geq 0$ ): By 2°, for each  $t \geq 0$  we have

$$\pi_+ \tau^t \mathcal{K} = \pi_+ \mathcal{X} (\pi_+ + \pi_-) \tau^t \mathcal{K}_0 = \mathcal{X} \pi_+ \tau^t \mathcal{K}_0 + \pi_+ \mathcal{X} \pi_- \tau^t \mathcal{K}_0 \quad (130)$$

$$= \mathcal{X} \mathcal{K}_0 \mathcal{A}_0^t - \mathcal{K} \mathcal{B} \tau^t \mathcal{K}_0 = \mathcal{K} (\mathcal{A}_0^t - \mathcal{B} \tau^t \mathcal{K}_0) = \mathcal{K} \mathcal{A}^t. \quad (131)$$

4° By 2°, we have  $-\mathcal{K} \mathcal{B} = \pi_+ \mathcal{X} \pi_-$  on  $L_\omega^2$ , by density (See Theorem B.3.11 of [M02]). From this and 3° we observe that Definition 2.1 is satisfied.  $\square$

In Theorem 7.2 we gave six equivalent conditions for the IRE. Now we shall prove them and give a (partial) seventh one:

**Lemma 10.7** ( $\hat{\mathcal{S}} = \hat{\mathcal{X}}^* S \hat{\mathcal{X}} \Leftrightarrow [\mathcal{K} \mid \mathcal{F}]$ ) *Theorem 7.2 holds. Moreover, a solution of (ii) is a solution (viii). Conversely, a solution of (viii) is a solution of (ii) if, e.g.,  $\mathcal{N}$  and  $\mathcal{X}^{-1}$  are q.r.c. and  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$ .*

(viii) *There are  $\mathcal{X} \in \mathcal{GTIC}_\infty(U)$ ,  $S \in \mathcal{B}(U)$  s.t. for all  $u \in L_c^2(\mathbb{R}_+; U)$  we have  $\mathcal{N}u \in L^2$  (here  $\mathcal{N} := \mathcal{D}\mathcal{X}^{-1}$ ), and*

$$(\hat{\mathcal{N}}u_0)^* J(\hat{\mathcal{N}}u_0) = S \quad \text{a.e. on } i\mathbb{R} \quad (u_0 \in U). \quad (132)$$

In (132),  $\hat{\mathcal{N}}u_0$  denotes the boundary function of  $\hat{\mathcal{N}}u_0 \in H^2(\mathbb{C}^+; Y)$ . If  $\hat{\mathcal{D}}, \hat{\mathcal{X}} \in H^\infty$  (or if  $\sigma(A) \cap \mathbb{C}^+$  is at most countable), then (132) is equivalent to  $\hat{\mathcal{D}}^* J \hat{\mathcal{D}} = \hat{\mathcal{X}}^* S \hat{\mathcal{X}}$  in  $L_{\text{strong}}^\infty(i\mathbb{R}; \mathcal{B}(U))$  (equivalently, a.e. on  $i\mathbb{R}$  for each  $u_0$ , not necessarily pointwise a.e. in  $\mathcal{B}(U)$  unless  $U$  is separable; see Chapter 3 of [M02] for details). Condition (132) is equivalent to

$$\langle \mathcal{N}u, J\mathcal{N}u \rangle = \langle u, Su \rangle \quad (u \in L_c^2(\mathbb{R}_+; U)), \quad (133)$$

by the proof of Lemma 10.4(c2).

Obviously, for any  $\mathcal{X} \in \mathcal{GTIC}_\infty(U)$ ,  $S \in \mathcal{GB}(U)$ , we get (46a) and (46c) (and (47a) and (47c)) from the  $\mathcal{S}^t$ -IRE (and  $\hat{\mathcal{S}}$ -IRE) by setting  $\mathcal{K} := \mathcal{X}\mathcal{K}_0$ ; the additional condition above is equivalent to the middle equation of the IRE (and  $\widehat{\text{IRE}}$ ) (which in turn can be used to show that  $[\mathcal{K} \mid \mathcal{F}]$  is an admissible state-feedback pair if  $\Sigma_0$  is a WPLS).

**Proof of Lemma 10.7:**  $1^\circ (vi) \Leftrightarrow (v) \Leftrightarrow (i) \Rightarrow (ii) \Leftrightarrow (viii)$ : The equivalence  $(vi) \Leftrightarrow (v) \Leftrightarrow (i) \Leftrightarrow (vii)$  follows from Theorem 10.1 and Lemma 10.5. For the rest, assume (i) (so that  $\Sigma_{\text{opt}} = \Sigma_\circ [I_0]$  in Theorem 4.7, by uniqueness). Then the IRE and the  $\widehat{\text{IRE}}$  hold and  $S$  is one-to-one, by Theorem 10.1. Claim (ii) follows from (47b), and (viii) from Lemma 10.4(c2).

$2^\circ (ii) \Rightarrow (i)$ : Assume (ii). By Theorem 4.7, there is a unique  $J$ -optimal control  $\mathcal{K}_0$  in WPLS form. Apply Lemma 10.6 to obtain a solution of the IRE (with  $\mathcal{F} = I - \mathcal{X}$ ,  $\mathcal{K} = \mathcal{X}\mathcal{K}_0$ ).

By Lemma 10.6(b) and  $2.2^\circ$ ,  $[\frac{\mathcal{A}}{\mathcal{K}} \mid \frac{\mathcal{B}}{\mathcal{F}}]$  is a WPLS. Since  $\mathcal{X} \in \mathcal{GTIC}_\infty$ , this means that the pair  $[\mathcal{K} \mid \mathcal{F}]$  is an admissible state-feedback pair for  $\Sigma$ ; since  $\mathcal{K}_0 = \mathcal{X}^{-1}\mathcal{K}$ , we have  $\Sigma_0 = \Sigma_\circ [I_0]$  and the pair is  $J$ -optimal.

$2.2^\circ S$  is one-to-one: As noted below Lemma 9.6, the  $\mathcal{S}^t$ -IRE equals the DARE, hence  $\mathcal{S}^t$  is one-to-one, by the discretized Theorem 9.9.1(f2) of [M02], hence so is  $S$ , by (46b).

$3^\circ (iii) \Leftrightarrow (ii)$ : This follows from Lemma 10.6 (with  $\Sigma_0 := \Sigma_{\text{opt}}$ ).

$4^\circ (ii) \Rightarrow (iv)$ : Assume (ii) (on  $\mathbb{C}_\alpha^+$ ; the claim on  $\mathbb{C}_\alpha^+ \setminus \mathbb{C}_\beta^+$  follows from  $2^\circ$ , which is otherwise unnecessary). Increase  $\alpha$  if necessary (from  $1^\circ$  and the  $\vartheta$ -stability of  $\mathcal{K}_0$  we could deduce that any  $\alpha > \max\{\omega_A, \vartheta\}$  will do) to have  $\mathcal{D}_+, \mathcal{X}_+, \mathcal{X}_+^{-1} \in \text{TIC}_{-\delta}$  for some  $\delta > 0$ , where  $\mathcal{X}_+ := e^{-\alpha} \mathcal{X} e^\alpha$  (i.e.,  $\widehat{\mathcal{X}_+}(s) = \widehat{\mathcal{X}}(s + \alpha)$ ). Then  $\widehat{\mathcal{X}_+}^* S \widehat{\mathcal{X}_+} = \widehat{\mathcal{D}_+}^* J_+ \widehat{\mathcal{D}_+}$  on  $i\mathbb{R}$ , hence (iv) holds ( $S$  is one-to-one by (b)).

$6^\circ$  *The claims on (viii)*: By  $1^\circ$  above, (i) (hence also (ii) and (iii)) implies (viii). Assume then that (viii) holds and that  $\mathcal{N}$  and  $\mathcal{M} := \mathcal{X}^{-1}$  are q.r.c.

$6.1^\circ$  It obviously follows that  $\mathcal{M}[L^2(\mathbb{R}_+; U)] \subset \mathcal{U}_{\text{out}}(0) \subset \mathcal{M}[L^2(\mathbb{R}_+; U)]$ .

$6.2^\circ$  We have  $\pi_+ \mathcal{X} \pi_- = \mathcal{K} \mathcal{B}$ , where  $\mathcal{K} := \mathcal{X}\mathcal{K}_0$ : Let  $u \in L^2([-T, 0); U)$ ,  $T > 0$ ,  $v \in \mathcal{U}_{\text{out}}(0)$ , so that  $v_\circ := \mathcal{X}v$ ,  $f := \pi_{[0, T)} \mathcal{X} \tau^{-T} u \in L^2(\mathbb{R}_+; U)$ . Set  $\mathcal{T} := \pi_+ \mathcal{M} \pi_- \mathcal{X}$  to obtain that

$$\langle \mathcal{D}(u + \mathcal{T}u), J\mathcal{D}v \rangle = \langle \mathcal{N} \pi_- \mathcal{X}u, J\mathcal{N}v_\circ \rangle = \langle \mathcal{N}f, J\mathcal{N} \tau^{-T} v_\circ \rangle \quad (134)$$

$$= (2\pi)^{-1} \langle \hat{\mathcal{N}} \hat{f}, J \hat{\mathcal{N}} \tau^{-T} \widehat{v_\circ} \rangle \quad (135)$$

$$= (2\pi)^{-1} \langle \hat{f}, S \tau^{-T} \widehat{v_\circ} \rangle = \langle f, S \tau^{-T} v_\circ \rangle = 0, \quad (136)$$

since  $\tau^{-T} v_\circ$  is supported on  $[T, +\infty)$ . Given  $u \in L_c^2(\mathbb{R}_-; U)$ , we have  $\langle \pi_+ \mathcal{D}(u + \mathcal{T}u), \mathcal{D}v \rangle = 0$  for all  $v \in \mathcal{U}_{\text{out}}(0)$ . But  $\pi_+ \mathcal{D} = \mathcal{C} \mathcal{B}u$ , hence  $\mathcal{T}u$  must be the unique  $J$ -optimal control for  $x_0 := \mathcal{B}u$ , i.e.,  $\mathcal{T}u = \mathcal{K}_\circ \mathcal{B}u$  (we have  $\mathcal{T}u \in \mathcal{U}_{\text{out}}(x_0)$ , because  $\mathcal{T}u \subset \mathcal{M} L_c^2 \subset L^2$

and  $\mathcal{C}x_0 + \mathcal{D}\mathcal{T}u = \mathcal{N}\pi_- \mathcal{X}u \subset \mathcal{N}L_c^2 \subset L^2$ , because  $\pi_- \mathcal{X}u \in L_c^2$ ). Consequently,  $\mathcal{K}\mathcal{B}u = \mathcal{X}\mathcal{T}u = -\pi_+ \mathcal{X}\pi_- u$  (because  $\mathcal{T} = \pi_+ \mathcal{M}\pi_- \mathcal{X}\pi_- = \pi_+ I \pi_- - \pi_+ \mathcal{M}\pi_+ \mathcal{X}\pi_- = -\pi_+ \mathcal{M}\pi_+ \mathcal{X}\pi_-$ ).

6.3° *Claim (i) holds:* Set  $\mathcal{F} := I - \mathcal{X}$ , so that  $\pi_+ \mathcal{F}\pi_- = \mathcal{K}\mathcal{B}$  (on  $L_c^2$ , hence on  $L^2$  for  $\omega$  big enough, by continuity), and  $\mathcal{K}\mathcal{A}^t = \mathcal{X}\mathcal{K}_0(\mathcal{A}_0^t - \mathcal{B}^t \mathcal{K}_0) = \dots = \pi_+ \tau^t \mathcal{K}$  (see (8.56) of [M02]), hence  $\left[\frac{\mathcal{A}}{\mathcal{X}} \middle| \frac{\mathcal{B}}{\mathcal{F}}\right]$  is a WPLS. Obviously, by using  $[\mathcal{K} \mid \mathcal{F}]$  for  $\Sigma$ , we get  $\Sigma_0 = \Sigma_{\cup} \begin{bmatrix} I \\ 0 \end{bmatrix}$ .

*Remarks:* 1. A similar claim holds for any  $\mathcal{U}_* \subset \mathcal{U}_{\text{out}}$ .  
2. By Example 9.13.2 of [M02], condition (viii) is not sufficient without an additional assumption connecting  $\mathcal{M}$  to  $\mathcal{U}_*$ , to  $\mathcal{P}$  or to  $\mathcal{K}_{\cup}$ .

5° (iv) $\Rightarrow$ (i): Define “the extended shifted systems”  $\Sigma_+$  and  $\Sigma_{\text{opt}}^+$  as follows:

$$\left[ \begin{array}{c|c} \mathcal{A}_+ & \mathcal{B}_+ \\ \hline \mathcal{C}_+ & \mathcal{D}_+ \end{array} \right] := \left[ \begin{array}{c|c} e^{-\alpha} \mathcal{A} & \mathcal{B}e^{\alpha} \\ \hline e^{-\alpha} \mathcal{C} & e^{-\alpha} \mathcal{D}e^{\alpha} \\ e^{-\alpha} \mathcal{A} & e^{-\alpha} \mathcal{B}\tau e^{\alpha} \end{array} \right], \quad \left[ \begin{array}{c} \mathcal{A}_{\text{opt}}^+ \\ \hline \mathcal{C}_{\text{opt}}^+ \\ \hline \mathcal{K}_{\text{opt}}^+ \end{array} \right] := e^{-\alpha} \cdot \left[ \begin{array}{c} \mathcal{A}_{\text{opt}} \\ \hline \mathcal{C}_{\text{opt}} \\ \hline \mathcal{A}_{\text{opt}} \\ \hline \mathcal{K}_{\text{opt}} \end{array} \right] \quad (137)$$

(These two systems equal  $\Sigma$  and  $\Sigma_{\text{opt}}$  with the third row added and  $A$  replaced by  $A - \alpha$  and  $A_{\text{opt}}$  by  $A_{\text{opt}} - \alpha$ , by Lemma 6.2.9(c) of [M02].) These systems are exponentially stable, since  $-\alpha, \omega_A - \alpha < 0$ . Set  $r := \alpha$ . Since  $\Sigma_{\text{opt}}$  is  $J$ -optimal and  $\mathcal{P} = \mathcal{C}_{\text{opt}}^* J_+ \mathcal{C}_{\text{opt}}$ , equations (57) and (58) hold, by Theorem 9.1(f). But (57) is exactly  $\mathcal{P} = (\mathcal{C}_{\text{opt}}^+)^* J \mathcal{C}_{\text{opt}}^+$ , and (58) is exactly  $0 = (\mathcal{C}_{\text{opt}}^+)^* J_+ \mathcal{D}_+$ , which means that  $\Sigma_{\text{opt}}^+$  is  $J_+$ -optimal for  $\Sigma_+$  over  $\mathcal{U}_{\text{exp}}^{\Sigma_+} = \mathcal{U}_{\text{out}}^{\Sigma_+}$  (by Theorem 9.1(c)). (By 5.1°, it is the only one.)

5.1° *Uniqueness over  $\mathcal{U}_{\text{out}}^{\Sigma_+}(x_0)$ :* By Lemma 4.4(ii), a control  $u \in \mathcal{U}_{\text{out}}^{\Sigma_+}(0)$  is  $J_+$ -optimal for 0 iff  $0 = \langle \mathcal{D}_+ \eta, J_+ \mathcal{D}_+ u \rangle = \langle \mathcal{X}_+ \eta, S \mathcal{X}_+ u \rangle$  ( $\eta \in L^2(\mathbb{R}_+; U)$ ) (recall that  $\mathcal{D}_+^* J_+ \mathcal{D}_+ = \mathcal{X}_+^* S \mathcal{X}_+$ ), i.e., iff  $\mathcal{X}_+^* S \mathcal{X}_+ u = 0$ . Since  $S$  is one-to-one, this implies that  $u = 0$ . By Lemma 4.4(c), it follows that there is at most one  $J_+$ -optimal control over  $\mathcal{U}_{\text{out}}^{\Sigma_+}(x_0)$  for each  $x_0 \in H$ .

5.2° Set  $\mathcal{M}_+ := \mathcal{X}_+^{-1} \in \mathcal{GTIC}(U)$ ,  $\mathcal{N}_+ := \mathcal{D}_+ \mathcal{M}_+$ . Trivially,  $\mathcal{M}_+ u \in L^2 \Rightarrow u = \mathcal{X}_+ \mathcal{M}_+ u \in L^2$ , hence we can apply condition (viii) of Lemma 10.7 to  $\Sigma_+$  (recall that (133) implies (132) and note that “(viii) $\Rightarrow$ (i)” was established above in 6°) to obtain  $(\Sigma_{\text{opt}})_+$  in state-feedback form (i.e., a  $J_+$ -optimal pair  $[\mathcal{K}_+ \mid \mathcal{F}_+]$  over  $\mathcal{U}_{\text{out}}^{\Sigma_+}$ ; by 5.1°, we must have  $(\Sigma_+)_{\cup} \begin{bmatrix} I \\ 0 \end{bmatrix} = (\Sigma_{\text{opt}})_+$ ). Set  $\mathcal{K} := e^{\alpha} \mathcal{K}_+$ ,  $\mathcal{F} := e^{\alpha} \mathcal{F}_+ e^{-\alpha}$  to obtain  $\Sigma_{\text{opt}}$  in state-feedback form, i.e., a  $J$ -optimal state-feedback pair for  $\Sigma$ .

*Remark:* Assume (iv). Apply (112) to  $\Sigma_+$  and let  $t \rightarrow +\infty$  to obtain that

$$S \mathcal{K}_+ = -\pi_+ (\mathcal{N}_+)^* J_+ \mathcal{C}_+ = -\pi_+ e^{\alpha} \cdot \left[ \begin{array}{c} \mathcal{N} \\ \hline \mathcal{B}_{\cup} \mathcal{T} \end{array} \right]^* e^{-2\alpha} \cdot \left[ \begin{array}{c} J \mathcal{C} \\ \hline 2\alpha \mathcal{P} \mathcal{A} \end{array} \right]. \quad (138)$$

Since  $\mathcal{K} = e^{\alpha} \mathcal{K}_+$ , this determines also  $\mathcal{K}$  uniquely (recall from (b) that  $S$  is one-to-one) modulo the constant  $E$  mentioned in (b).

(a) This follows from 1°–5° above (for any  $\alpha > \max\{\vartheta, \omega_A\}$ ).

(b) This follows from Theorem 10.1.  $\square$

**Notes for Section 10:** We defined the IRE and presented the corresponding theory in Theorem 9.9.1 of [M02]; that contained Theorem 10.1 except for (iii). Lemmas 10.3 and 10.4 are from [M02], and many of the computations for the latter are from [S98b] (see the notes for Section 7). Otherwise the results seem to be new.

## 11 $J$ -coercivity

In this section, we present the (generalization to WPLSs of)  $J$ -coercivity, the standard coercivity condition for control problems, and derive results that lead to the theory of Section 5.

As explained before Theorem 4.6,  $J$ -coercivity means that the *Popov Toeplitz operator*  $\mathcal{S}_{\text{PT}} := \mathcal{D}^* J \mathcal{D}$  is boundedly invertible  $\mathcal{U}_*(0) \rightarrow \mathcal{U}_*(0)^*$  (actually,  $\mathcal{S}_{\text{PT}} = \pi_+ \mathcal{D}^* J \mathcal{D} \pi_+$ , but the condition remains the same since  $\mathcal{U}_*(0) \subset L_{\mathcal{D}}^2(\mathbb{R}_+; U)$ ). If (f)  $\mathcal{J}(0, u) \geq 0$  ( $u \in$

$\mathcal{U}_*(0)$ ), then a control is minimizing iff it is  $J$ -optimal; moreover, then  $J$ -coercivity is equivalent to the existence of  $\epsilon > 0$  s.t.

$$\mathcal{J}(0, u) \geq \epsilon \|u\|_{\mathcal{U}_*}^2 \quad (u \in \mathcal{U}_*(0)). \quad (139)$$

In the general (indefinite) case, the above condition becomes more complicated (see (v)) but still nicely applicable to  $H^\infty$  control problems (see Chapter 11 of [M02]):

**Lemma 11.1 (J-coercivity)** *The following are equivalent:*

- (i)  $\mathcal{D}$  is  $J$ -coercive;
- (ii)  $\mathcal{D}^* J \mathcal{D} \in \mathcal{B}(\mathcal{U}_*(0), \mathcal{U}_*(0)^*)$  is coercive;
- (iii)  $\mathcal{D}^* J \mathcal{D} \in \mathcal{B}(\mathcal{U}_*(0), \mathcal{U}_*(0)^*)$  is (boundedly) invertible;
- (iv)  $\mathcal{D}|_{\mathcal{U}_*(0)}$  and  $\tilde{J}|_{\mathcal{D}[\mathcal{U}_*(0)]}$  are coercive;
- (v) There is  $\epsilon > 0$  s.t. for all nonzero  $u \in \mathcal{U}_*(0)$  there is a nonzero  $v \in \mathcal{U}_*(0)$  s.t.

$$\langle \mathcal{D}v, J \mathcal{D}u \rangle_{L^2} \geq \epsilon \|u\|_{\mathcal{U}_*} \|v\|_{\mathcal{U}_*}. \quad (140)$$

Moreover, if (ii) holds (and  $\mathcal{U}_*(0) \neq \{0\}, \emptyset$ ), then  $\epsilon = \|\mathcal{S}_{PT}^{-1}\|_{\mathcal{B}(\mathcal{U}_*(0)^*, \mathcal{U}_*(0))}$  is the maximal value of  $\epsilon$  in (v), and  $\vartheta \geq 0$ .

Note from Lemma 4.4(a) that  $\|u\|_{\mathcal{U}_{out}}$  is equivalent to  $\max\{\|u\|_2, \|\mathcal{B}\tau u\|_2\}$  and  $\|u\|_{\mathcal{U}_{exp}}$  to  $\max\{\|u\|_2, \|\mathcal{B}\tau u\|_2\}$ .

**Proof:** (A linear map  $D : X \rightarrow Y$  is *coercive* iff there is  $\epsilon > 0$  s.t.  $\|Dx\| \geq \epsilon \|x\|$  ( $x \in X$ )). In (ii) and (iii), the symbol  $\mathcal{D}^*$  refers to the adjoint of  $\mathcal{D}|_{\mathcal{U}_*(0)}$ , hence  $\langle v, \mathcal{D}^* J \mathcal{D}u \rangle := \langle \mathcal{D}v, J \mathcal{D}u \rangle$ .

1° “(i)  $\Leftrightarrow$  (iii)  $\Rightarrow$  (ii)  $\Leftrightarrow$  (v)  $\Leftarrow$  (iv)”: In Theorem 4.6 we used (iii) as the definition. Obviously, (ii) follows from (iii), and (ii) is equivalent to (v). Similarly, (iv) implies (v) ( $\langle \mathcal{D}v, J \mathcal{D}u \rangle \geq \epsilon' \|\mathcal{D}v\| \|\mathcal{D}u\| \geq \epsilon' (\epsilon'')^2 \|u\|_{\mathcal{U}_*} \|v\|_{\mathcal{U}_*}$ ).

2° “(ii)  $\Rightarrow$  (iii)  $\Leftrightarrow$  (iv)”: Define  $\mathcal{Y} := \mathcal{D}[\mathcal{U}_*(0)] \subset L^2(\mathbb{R}_+; Y)$ , let  $P$  be the orthogonal projection  $L^2(\mathbb{R}_+; Y) \rightarrow \mathcal{Y}$ ,  $\tilde{J} := PJP^* \in \mathcal{B}(\mathcal{Y})$ ,  $D \in \mathcal{B}(\mathcal{U}_*(0), \mathcal{Y})$ , so that  $D^* \in \mathcal{B}(\mathcal{Y}, \mathcal{U}_*(0)^*)$ . Assume (ii), i.e., that  $D^* \tilde{J} D$  is coercive. Then so are  $D$  and  $\tilde{J}$ , hence  $\mathcal{Y} = \mathcal{Y}$  and  $D \in \mathcal{B}(\mathcal{U}_*(0), \mathcal{Y})$  is an isomorphism onto, hence invertible. Being self-adjoint and coercive, also  $\tilde{J}$  is invertible (see A.3.5(c2) and A.3.4(N5) of [M02]). Thus, (iii) and (iv) hold.

(Note: by the above,  $\mathcal{U}_*(0)$  is a Hilbert space (and can thus be identified with its dual when (i) holds; of course,  $\mathcal{U}_{exp}$  and  $\mathcal{U}_{out}$  have natural inner products even without  $J$ -coercivity.)

3° On  $\epsilon = \|\mathcal{S}_{PT}^{-1}\|$ : Obviously,  $\inf_{u \neq 0} \|\mathcal{S}_{PT} u\| / \|u\| = \|\mathcal{S}_{PT}^{-1}\|$ , hence  $\epsilon$  cannot be any larger. Conversely, there is  $\Lambda \in \mathcal{U}_*(0)^{**}$  s.t.  $\|\Lambda\| \leq 1$  and  $\Lambda \mathcal{S}_{PT} u = \|\mathcal{S}_{PT} u\|_{\mathcal{U}_*(0)^*}$ . By reflexivity (which obviously follows from (iii)), we have  $\Lambda = \langle v, \cdot \rangle$  for some  $v \in \mathcal{U}_*(0)$ . Obviously,  $\|v\| = \|\Lambda\| = 1$ . Thus,  $\langle v, \mathcal{S}_{PT} u \rangle = \|\mathcal{S}_{PT} u\| \geq \epsilon \|u\| \|v\|$  for  $\epsilon := \|\mathcal{S}_{PT}^{-1}\|$ .

4°  $\vartheta \geq 0$ : Whenever  $\mathcal{U}_*(0) \neq \{0\}$  and  $\|\mathcal{D}u\|_2 \geq \epsilon \|u\|_{L_\vartheta^2}$  for some  $\epsilon > 0$  and each  $u \in \mathcal{U}_*(0)$  we have  $\vartheta = 0$ , because otherwise  $\|\mathcal{D}u\|_2 = \|\tau^{-t} \mathcal{D}u\|_2 = \|\mathcal{D} \tau^{-t} u\|_2 \geq \epsilon \|\tau^{-t} u\|_{L_\vartheta^2} = \epsilon e^{t\vartheta} \|u\|_{L_\vartheta^2} \rightarrow +\infty$ , as  $t \rightarrow +\infty$  (we have  $\tau^{-t} u \in \mathcal{U}_*(0)$  ( $t \geq 0$ ), by Lemma 9.3).  $\square$

Many special cases of  $J$ -coercivity are commonly used in the study of finite-dimensional systems, Pritchard–Salamon systems or other special cases of WPLSs. Therefore, we now recall from [M02] that for, e.g., systems having smoothing semigroups or bounded input operators,  $I$ -coercivity over  $\mathcal{U}_{exp}$ , i.e., condition

$$(i) \quad \|\mathcal{D}u\|_2 \geq \epsilon (\|u\|_2 + \|\mathcal{B}\tau u\|_2) \quad (u \in \mathcal{U}_{exp}(0)),$$

is equivalent to classical coercivity assumptions:

**Theorem 11.2** ( $\mathcal{S}_{PT} \gg 0$ ) *Assume that  $J \gg 0$  and that the state-FCC is satisfied. (a) If  $B \in \mathcal{B}(U, H)$ , then also any of (ii)–(vii) is equivalent to (positive)  $J$ -coercivity over  $\mathcal{U}_{exp}$ :*

$$(ii) \quad D^* D \gg 0, \text{ and } \|\mathcal{D}u\|_2 \geq \epsilon \|\mathcal{B}\tau u\|_2 \text{ for some } \epsilon > 0 \text{ and all } u \in \mathcal{U}_{exp}(0);$$



- (iii)  $(ir - A)x_0 = Bu_0 \implies \|C_w x_0 + Du_0\|_Y \geq \epsilon(\|x_0\|_H + \|u_0\|_U)$  for some  $\epsilon > 0$  and all  $x_0 \in H$ ,  $u_0 \in U$ ,  $r \in \mathbb{R}$ ;
- (iv)  $D^*D \gg 0$ , and  $(ir - A)x_0 = Bu_0 \implies \|C_w x_0 + Du_0\|_Y \geq \epsilon\|x_0\|_H$  for some  $\epsilon > 0$  and all  $x_0 \in H$ ,  $u_0 \in U$ ,  $r \in \mathbb{R}$ ;
- (v)  $\| \begin{bmatrix} A-ir & B \\ C_w & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \|_{H \times Y} \geq \epsilon \| \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \|_{H \times U}$  for some  $\epsilon > 0$  and all  $r \in \mathbb{R}$ ,  $x_0 \in H$ ,  $u_0 \in U$ ;
- (vi)  $D^*D \gg 0$ , and there is a unique minimizing  $u \in \mathcal{U}_{\text{exp}}(x_0)$  for each  $x_0 \in H$ ;
- (vii)  $D^*D \gg 0$ , and the  $B_w^*$ -ARE (p. 33) has an exponentially stabilizing solution.

(b) If  $\mathcal{A}B \in L^1([0, 1]; \mathcal{B}(U, H))$ ,  $C \in \mathcal{B}(H, Y)$  and  $(D^*JC = 0$  or  $D^*JD \in \mathcal{GB}(U))$ , then (i)–(vii) are still equivalent (in (vii) we must have  $B_w^*$  in place of  $B^*$  and require that  $\mathcal{P}[H] \subset \text{Dom}(B_w^*)$ ).

(c) Assume that  $\mathcal{D}$  is ULR. If  $B$  is not maximally unbounded or  $\mathcal{A}B \in L^1([0, 1]; \mathcal{B}(U, H))$ , then (i)–(v) are equivalent (and imply (vi)).

(The proof is given on p. 65. Condition (v) is called “no invariant zeros”. If  $\|(A - ir)x_0 + Bu_0\|_H < \infty$ , then  $x_0 \in \text{Dom}(C_w)$  (since here  $\mathcal{D}$  is regular), as noted below Definition 2.6. See Proposition 10.3.2 of [M02] for more general systems and results.)

Similarly,  $I$ -coercivity over  $\mathcal{U}_{\text{out}}$  (i.e.,  $\|\mathcal{D}u\|_2 \geq \epsilon\|u\|_2$  ( $u \in \mathcal{U}_{\text{out}}(0)$ )) is a generalization of several classical assumptions, such as “no transmission zeros” (Proposition 10.3.1(a) of [M02]).

Next we prove Theorems 4.6 and 4.7:

**Proof of Theorem 4.6:** (From that of Theorem 8.2.5 of [M02].)

1° *J-optimal control:* (We use the results and notation of the proof of Lemma 11.1, in particular, we identify  $\mathcal{U}_*(0)^*$  with  $\mathcal{U}_*(0)$ .) Let  $x_0 \in H$ ,  $\tilde{u} \in \mathcal{U}_*(x_0)$ . Set  $\tilde{y} := \mathcal{C}x_0 + \mathcal{D}\tilde{u}$ ,  $v := -(D^*\tilde{J}D)^{-1}D^*\tilde{J}\tilde{y} \in \mathcal{U}_*(0)$ ,  $u := \tilde{u} + v \in \mathcal{U}_*(x_0)$ ,  $y := \mathcal{C}x_0 + \mathcal{D}u$ . Then

$$\langle y, JD\eta \rangle_{L^2} = \langle \tilde{y} + Dv, \tilde{J}D\eta \rangle_{L^2} = \langle D^*\tilde{J}\tilde{y} + D^*\tilde{J}Dv, \eta \rangle_{\mathcal{U}_*(0)} = 0 \quad (141)$$

for all  $\eta \in \mathcal{U}_*(0)$ , hence  $u$  is  $J$ -optimal for  $x_0$ .

2° *Uniqueness:* The difference of two  $J$ -optimal controls for any  $x_0$  is  $J$ -optimal for 0, hence we can assume that  $x_0 = 0$ . If  $u$  is  $J$ -optimal for  $x_0 = 0$ , then  $\langle Dv, JDv \rangle = 0$  for all  $v \in \mathcal{U}_*(0)$ , hence then  $\|u\|_{\mathcal{U}_*} = 0$ , by (v), hence  $u = 0$ .

3° *Case  $\mathcal{S}_{PT} \geq 0$ :* This follows from Lemma 4.4(iii).  $\square$

**Proof of Theorem 4.7:** (From that of Theorem 8.3.9 of [M02].)

1°  $\Sigma_{\text{opt}}$  is a WPLS: Let  $x_0 \in H$ ,  $t \geq 0$ . We first show that  $\pi_+\tau^t\mathcal{K}_{\text{opt}}x_0$  is  $J$ -optimal for  $\mathcal{A}_{\text{opt}}^t x_0$ , i.e., equal to  $\mathcal{K}_{\text{opt}}\mathcal{A}_{\text{opt}}^t x_0$ : For  $\eta \in \mathcal{U}_*(0)$  we have  $\tau^{-t}\eta \in \mathcal{U}_*(0)$ , hence

$$\langle J\pi_+\tau^t\mathcal{C}_{\text{opt}}x_0, \mathcal{D}\eta \rangle_{L^2} = \langle J\mathcal{C}_{\text{opt}}x_0, \mathcal{D}\tau^{-t}\eta \rangle_{L^2} = 0 \quad (\eta \in \mathcal{U}_*(0)). \quad (142)$$

But

$$\pi_+\tau^t\mathcal{C}_{\text{opt}}x_0 = \pi_+\tau^t(\mathcal{C}x_0 + \mathcal{D}\mathcal{K}_{\text{opt}}x_0) = \mathcal{C}\mathcal{A}^t x_0 + \pi_+\mathcal{D}(\pi_+ + \pi_-)\tau^t\mathcal{K}_{\text{opt}}x_0 \quad (143)$$

$$= \mathcal{C}\mathcal{A}^t x_0 + \mathcal{D}\pi_+\tau^t\mathcal{K}_{\text{opt}}x_0 + \mathcal{C}\mathcal{B}\tau^t\mathcal{K}_{\text{opt}}x_0 = \mathcal{C}\mathcal{A}_{\text{opt}}^t x_0 + \mathcal{D}\pi_+\tau^t\mathcal{K}_{\text{opt}}x_0. \quad (144)$$

This and (142) imply that  $\pi_+\tau^t\mathcal{K}_{\text{opt}}x_0$  is  $J$ -optimal for  $\mathcal{A}_{\text{opt}}^t x_0$ ; thus

$$\pi_+\tau^t\mathcal{K}_{\text{opt}}x_0 = u_{\text{opt}}(\mathcal{A}_{\text{opt}}^t x_0) = \mathcal{K}_{\text{opt}}\mathcal{A}_{\text{opt}}^t x_0, \quad \pi_+\tau^t\mathcal{C}_{\text{opt}}x_0 = y_{\text{opt}}(\mathcal{A}_{\text{opt}}^t x_0) = \mathcal{C}_{\text{opt}}\mathcal{A}_{\text{opt}}^t x_0. \quad (145)$$

By the dynamic programming principle,  $\mathcal{A}$  is a semigroup; a detailed proof of this fact goes as follows, using (145):

$$\mathcal{A}_{\text{opt}}^s \mathcal{A}_{\text{opt}}^t = \mathcal{A}^s(\mathcal{A}^t + \mathcal{B}\tau^t\mathcal{K}_{\text{opt}}) + \mathcal{B}\tau^s\mathcal{K}_{\text{opt}}\mathcal{A}_{\text{opt}}^t \quad (146)$$

$$= \mathcal{A}^s \mathcal{A}^t + \mathcal{B}\tau^s\pi_-\tau^t\mathcal{K}_{\text{opt}} + \mathcal{B}\tau^s\pi_+\tau^t\mathcal{K}_{\text{opt}} = \mathcal{A}^s \mathcal{A}^t + \mathcal{B}\tau^{s+t}\mathcal{K}_{\text{opt}} = \mathcal{A}_{\text{opt}}^{t+s}. \quad (147)$$

Obviously,  $\mathcal{A}_{\text{opt}}^0 = \mathcal{A}^0 = I$ , and  $t \mapsto \pi_-\tau^t u$  is continuous  $\mathbb{R}_+ \rightarrow L_\omega^2$  for any  $u \in L_{\text{loc}}^2$ , hence  $\mathcal{A}_{\text{opt}}x_0 = x_{\text{opt}}(x_0)$  is continuous for each  $x_0 \in H$ . Therefore,  $\mathcal{A}_{\text{opt}}$  is a  $C_0$ -semigroup. This and (145) imply that  $\Sigma_{\text{opt}}$  is a WPLS.

2° *The rest:* The claims on  $\mathcal{P}$  are obvious. The continuity of  $\mathcal{C}_{\text{opt}}$  (and  $\mathcal{K}_{\text{opt}} : H \rightarrow L^2_{\vartheta}(\mathbb{R}_+; U)$ ) follows from the closed-graph theorem and the exponential stability of  $\Sigma_{\text{opt}}$  from that of  $\mathcal{A}_{\text{opt}}$  (if  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$ ). (See Theorem 8.3.9 of [M02] for details and further results.)  $\square$

We shall soon need the following simple fact:

**Lemma 11.3** *Assume that  $0 \leq T \in \mathcal{GB}(H)$  and set  $\epsilon := \|T^{-1}\|^{-1}$ . Then  $T \geq \epsilon I$ .*  $\square$

(See Lemma A.3.1(b1') of [M02] or use a spectral decomposition or square root of  $T$ .)

Naturally, the FCC is necessary for  $\mathcal{P}$ ,  $\Sigma_{\text{opt}}$  and  $\mathcal{S}^t$  to exist. Next we show that if the FCC holds and  $\mathcal{S}_{\text{PT}} \in \mathcal{GB}$ , then also the “truncated Popov operators”  $\mathcal{S}^t := \mathcal{D}^{t*} J \mathcal{D}^t + \mathcal{B}^{t*} \mathcal{P} \mathcal{B}^t$  are invertible, with a uniform (over  $t$ ) norm bound for  $(\mathcal{S}^t)^{-1}$  on  $\mathcal{B}(L^2)$  (when  $\vartheta = 0$ , as in the case of  $\mathcal{U}_{\text{exp}}, \mathcal{U}_{\text{out}}$ ):

**Lemma 11.4** ( $\mathcal{S}_{\text{PT}} \in \mathcal{GB} \Rightarrow \mathcal{S}^t \in \mathcal{GB}$ ) *Assume that  $\mathcal{S}_{\text{PT}} \in \mathcal{GB}$  and that  $\mathcal{U}_*(x_0) \neq \emptyset \forall x_0 \in H$ .*

(a) *Then  $\mathcal{S}^t \in \mathcal{GB}(L^2_{\omega}([0, t]; U))$  for all  $\omega \in \mathbb{R}$ ,  $t > 0$ , and there are  $M_{\omega, t} < \infty$  s.t.*

$$\|(\mathcal{S}^t)^{-1}\|_{\mathcal{B}(L^2_{\vartheta}([0, t]; U), L^2_{-\vartheta}([0, t]; U))} \leq \|\mathcal{S}_{\text{PT}}^{-1}\|, \quad (148)$$

$$\|(\mathcal{S}^t)^{-1}\|_{\mathcal{B}(L^2_{\omega}([0, t]; U))} \leq M_{\omega, t} \|\mathcal{S}_{\text{PT}}^{-1}\| \quad (t > 0, \omega \in \mathbb{R}). \quad (149)$$

(b) *If  $\vartheta = 0$  and  $\mathcal{J}(0, \cdot) \geq 0$ , then  $\hat{\mathcal{S}}(s, s) \geq \epsilon I$  for  $s \in \mathbb{C}_{\omega_0}^+$ , where  $\omega_0 := \max\{0, \omega_A\}$  and  $\epsilon := \|\mathcal{S}_{\text{PT}}^{-1}\|^{-1} > 0$ .*

(c) *If  $\mathcal{J}(0, \cdot) \geq 0$ , then  $\hat{\mathcal{S}}(s, s) \geq 0$  for  $s \in \mathbb{C}_{\omega_0}^+$ .*

All results in Section 5 are based on Theorem 5.1, which is a corollary of (b) (i.e., of “ $\mathcal{S}_{\text{PT}} \gg 0 \Rightarrow \hat{\mathcal{S}}(s, s) \geq \epsilon I$ ”) and of Theorem 7.2.

**Proof:** W.l.o.g., we assume that  $U \neq \{0\}$ . Let  $\mathcal{K}_0$  be the (unique)  $J$ -optimal control in WPLS form. Let  $t > 0$ .

(a) Assume that  $u \in L^2([0, t]; U) \setminus \{0\}$ . Choose  $v$  for  $\epsilon := \|\mathcal{S}_{\text{PT}}^{-1}\|$  and  $P^t u$  as in (140). Since  $\|P^t u\|_{\mathcal{U}_*} \geq \|P^t u\|_{L^2_{\vartheta}} \geq \|u\|_{L^2_{\vartheta}([0, t]; U)}$ , we obtain from Lemma 9.8(a) that

$$\langle v, \mathcal{S}^t u \rangle_{L^2} = \langle v, \mathcal{S}_{\text{PT}} P^t u \rangle_{\mathcal{U}_*(0), \mathcal{U}_*(0)^*} \geq \epsilon \|P^t u\| \|v\|_{\mathcal{U}_*} \geq \epsilon \|u\|_{L^2_{\vartheta}([0, t]; U)} \|v\|_{L^2_{-\vartheta}([0, t]; U)}. \quad (150)$$

Since  $t, u, v$  were arbitrary, we get (148), which obviously implies (149).

(b) *We have  $\hat{\mathcal{S}}(s, s) \geq \epsilon I$ :* Let  $s \in \mathbb{C}_{\omega_0}^+$ , and set  $u(t) := e^{st} u_0$ , so that  $\pi_- u \in L^2 \cap L^2_{\omega}$ , and

$$(\mathcal{D}u)(t) = e^{st} \hat{\mathcal{D}}(s) u_0, \quad \mathcal{B} \tau^t u = e^{st} (s - A)^{-1} B u_0, \quad (t \in \mathbb{R}), \quad (151)$$

by Lemma 6.10 of [S98c]. By time-invariance (a similar computation was used for losslessness in Lemma 6.11 of [S98c]),

$$\langle \mathcal{D}^t \tau^{-t} u, J \mathcal{D}^t \tau^{-t} u \rangle = \langle \tau^{-t} \mathcal{D} \pi_{[-t, 0)} u, J \tau^{-t} \mathcal{D} \pi_{[-t, 0)} u \rangle = \int_{-\infty}^0 \langle \mathcal{D} \pi_{[-t, 0)} u, J \mathcal{D} \pi_{[-t, 0)} u \rangle_Y dr \quad (152)$$

$$\rightarrow \int_{-\infty}^0 \langle \mathcal{D} u, J \mathcal{D} u \rangle_Y(r) dr = \int_{-\infty}^0 e^{r(s+\bar{s})} \langle \hat{\mathcal{D}}(s) u_0, J \hat{\mathcal{D}}(s) u_0 \rangle_Y dr \quad (153)$$

$$= \langle \hat{\mathcal{D}}(s) u_0, J \hat{\mathcal{D}}(s) u_0 \rangle_Y / 2 \operatorname{Re} s, \quad (154)$$

as  $t \rightarrow +\infty$ , because  $\pi_- \mathcal{D} \pi_{(-\infty, t)} u \rightarrow 0$  in  $L^2_{\omega}$  (because  $\mathcal{D} \in \text{TIC}_{\omega}$  and  $\pi_- u \in L^2_{\omega}$ ), hence in  $L^2$  too (because  $\pi_- L^2_{\omega} \subset L^2$  continuously), for any  $\omega \in (\omega_0, \operatorname{Re} s)$ . Therefore,

$$\langle \tau^{-t} u, \mathcal{S}^t \tau^{-t} u \rangle \rightarrow \langle \hat{\mathcal{D}}(s) u_0, J \hat{\mathcal{D}}(s) u_0 \rangle_Y / 2 \operatorname{Re} s + \langle (s - A)^{-1} B u_0, \mathcal{P} (s - A)^{-1} B u_0 \rangle, \quad (155)$$

as  $t \rightarrow +\infty$ . But, by Lemma 11.3,  $\mathcal{S}^t \geq \epsilon I$  on  $L^2([0, t]; U)$ , hence

$$\langle \tau^{-t} u, \mathcal{S}^t \tau^{-t} u \rangle \geq \epsilon \int_0^t \|u_0\|^2 e^{2(r-t) \operatorname{Re} s} dr \rightarrow \epsilon \|u_0\|^2 / 2 \operatorname{Re} s. \quad (156)$$

Since  $u_0 \in U$  was arbitrary, we obtain from (155) and (156) that  $\hat{\mathcal{S}}(s, s) \geq \epsilon$ .

(c)  $\hat{\mathcal{S}}(s, s) \geq 0$ : The proof of (b) applies mutatis mutandis.  $\square$

**Notes for Section 11:** The important Lemma 11.4 seems to be completely new. Most of the rest we presented in [M02]. See p. 16 for further notes and Sections 8.4 and 10.3 of [M02] for further results.

## 12 Remaining proofs

In this section we give the remaining proofs, i.e., those on AREs and those for the theorems of Section 5. We start with two auxiliary lemmas.

If part of  $J$  is uniformly positive, the corresponding part of  $\mathcal{C}_\cup, \mathcal{D}_\cup$  becomes stable:

**Lemma 12.1** ( $(\mathcal{C}_2)_\cup, (\mathcal{D}_2)_\cup$  **are stable**) *Assume that  $\Sigma = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C}_1 & \mathcal{D}_1 \\ \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix}$ , and  $J = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} \geq 0$ ,  $J_{22} \gg 0$ . If the IRE has an admissible solution  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$  with  $\mathcal{P} \geq 0$ , then  $S \geq 0$ , and  $(\mathcal{C}_2)_\cup$  and  $(\mathcal{D}_2)_\cup$  are stable.*

**Proof:** (This is a variant of Proposition 10.7.3 of [M02]. Note that we have assumed that  $[\mathcal{C} \mid \mathcal{D}]$  and  $J$  have been split according to some split  $Y = Y_1 \times Y_2$ .)

Set  $\mathcal{N} := \mathcal{D}_\cup$ . By (111), for any  $t \geq 0$  we have  $\pi_{[0,t]} S = (\mathcal{N}_1^t)^* J_{11} \mathcal{N}_1^t + (\mathcal{N}_2^t)^* J_{22} \mathcal{N}_2^t + \mathcal{B}_\cup^t * \mathcal{P} \mathcal{B}_\cup^t \geq 0$ , hence  $S \geq 0$  and  $\|S\| \|u\|_2^2 \geq \|\pi_{[0,t]} J_{22}^{1/2} \mathcal{N}_2 u\|_2^2$  for all  $u \in L^2(\mathbb{R}_+; U)$ ,  $t \geq 0$ . Let  $t \rightarrow \infty$  to obtain that  $J_{22}^{1/2} \mathcal{N}_2$  is bounded  $L^2 \rightarrow L^2$ , hence so is  $\mathcal{N}_2 := (\mathcal{D}_2)_\cup$ .

Similarly, from (109) we observe that  $\mathcal{P} \geq [\pi_{[0,t]} (\mathcal{C}_2)_\cup]^* J_{22} \pi_{[0,t]} (\mathcal{C}_2)_\cup$ , hence  $(\mathcal{C}_2)_\cup$  is stable.  $\square$

By Theorem 10.1(e), the uniform positivity of the Popov operator implies that of the signature operator ( $\mathcal{S}_{PT} \gg 0 \Rightarrow S \gg 0$ ). We stated that the converse holds for  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$ ; in fact, it also holds for  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$  provided that the solution is  $(\mathcal{U}_*$ - and) I/O-stabilizing:

**Lemma 12.2** ( $\mathcal{S}_{PT} \gg 0 \Leftrightarrow S \gg 0 \Rightarrow$  **q.r.c.**) *Assume that the IRE has a  $\mathcal{U}_*$ -stabilizing solution  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$  with  $S \gg 0$ . Then (a)  $\mathcal{X} \in \mathcal{B}(\mathcal{U}_*(0), L^2(\mathbb{R}_+; U))$  and  $\mathcal{J}(0, u) = \langle \mathcal{X}u, S\mathcal{X}u \rangle \forall u \in \mathcal{U}_*(0)$ . Assume also that  $\mathcal{U}_* = \mathcal{U}_{\text{out}}$ . Then (b)  $[\mathcal{N}] v \in L^2 \Rightarrow v \in L^2$  (for all  $v \in L_{\text{loc}}^2(\mathbb{R}_+; U)$ ). Finally, if also  $\mathcal{N}, \mathcal{M} \in \text{TIC}$ , then (c)  $\mathcal{N}, \mathcal{M}$  are q.r.c.,  $\mathcal{S}_{PT} \gg 0$  and  $\mathcal{X} \in \mathcal{GB}(\mathcal{U}_{\text{out}}(0), L^2(\mathbb{R}_+; U))$ .*

This result was applied in Theorem 5.9.

**Proof:** (Recall that  $\mathcal{X} := I - \mathcal{F}$ ,  $\mathcal{M} := \mathcal{X}^{-1}$ ,  $\mathcal{N} := \mathcal{D}\mathcal{M}$ .)

(a)  $1^\circ$  Let  $v = \mathcal{X}u$ , where  $u \in \mathcal{U}_*(0)$ . Then, by (46b), we have

$$\langle v, S\pi_{[0,t]} v \rangle = \langle \mathcal{D}^t u, J \mathcal{D}^t u \rangle + \langle \mathcal{B}^t u, \mathcal{P} \mathcal{B}^t u \rangle. \quad (157)$$

Let  $t \rightarrow +\infty$  to observe that  $\int_0^\infty \langle v(t), S v(t) \rangle_U dt = \langle \mathcal{D}u, J \mathcal{D}u \rangle_{L^2} = \mathcal{J}(0, u)$ , by  $2^\circ$ . Since  $S \gg 0$ , we conclude that  $v \in L^2$ . Thus, (b) holds (note that  $\mathcal{J}(0, u) \leq \|J\| \|\mathcal{D}u\|_2^2 \leq \|J\| \|u\|_{\mathcal{U}_{\text{out}}}^2$ , hence  $\mathcal{X}$  is continuous).

$2^\circ$   $\langle \mathcal{B}^t u, \mathcal{P} \mathcal{B}^t u \rangle \rightarrow 0$ : Set  $x_0 := \mathcal{B}^t u$ ,  $\tilde{u} := \pi_+ \tau^t u \in \mathcal{U}_*(x_0)$  (Lemma 9.3). Then (recall 4. of Definition 2.1)

$$\langle \mathcal{C}x_0 + \mathcal{D}\tilde{u}, J - " - \rangle = \langle \mathcal{C}\mathcal{B}\tau^t u + \mathcal{D}\pi_+ \tau^t u, J - " - \rangle = \langle \pi_+ \mathcal{D}\pi_- \tau^t u + \pi_+ \mathcal{D}\pi_+ \tau^t u, J - " - \rangle \quad (158)$$

$$= \langle \pi_+ \mathcal{D}\tau^t u, J - " - \rangle = \langle \pi_+ \tau^t \mathcal{D}u, J - " - \rangle = \langle \pi_{[t,\infty)} \mathcal{D}u, J - " - \rangle. \quad (159)$$

But  $\langle x_0, \mathcal{P}x_0 \rangle$  is the minimum of  $\|\mathcal{C}x_0 + \mathcal{D}\tilde{u}\|_2^2$  over  $\tilde{u} \in \mathcal{U}_*(x_0)$ , hence  $\langle x_0, \mathcal{P}x_0 \rangle \leq \langle \mathcal{D}u, \pi_{[t,\infty)} J \mathcal{D}u \rangle \rightarrow 0$ , as  $t \rightarrow +\infty$ .

(b) Obviously,  $[\mathcal{N}] v \in L^2 \Leftrightarrow [\mathcal{D}] \mathcal{M}v \in L^2 \Leftrightarrow \mathcal{M}v \in \mathcal{U}_{\text{out}}(0)$ . By (a),  $\mathcal{M}v \in \mathcal{U}_{\text{out}}(0) \Rightarrow \mathcal{X}\mathcal{M}v \in L^2(\mathbb{R}_+; U)$ . But  $v = \mathcal{X}\mathcal{M}v$ .

(c) Now  $\mathcal{N}, \mathcal{M}$  are q.r.c., and we have  $v \in L^2 \Leftrightarrow [\mathcal{N}] v \in L^2 \Leftrightarrow \mathcal{M}v \in \mathcal{U}_{\text{out}}(0)$ , by (b). Consequently,  $\mathcal{U}_{\text{out}}(0) = \mathcal{M}[L^2(\mathbb{R}_+; U)]$ , and  $\mathcal{X} : \mathcal{U}_{\text{out}}(0) \rightarrow L^2(\mathbb{R}_+; U)$  is thus (boundedly) invertible. Let  $S \geq \epsilon I$ ,  $\epsilon > 0$ . By the proof of (a) and the above, we have

$$\langle u, \mathcal{S}_{PT} u \rangle := \mathcal{J}(0, u) = \langle \mathcal{X}u, S\mathcal{X}u \rangle \geq \epsilon \|\mathcal{X}u\|_2^2 \geq \epsilon \epsilon' \|u\|_{\mathcal{U}_{\text{out}}}^2, \quad (160)$$

i.e.,  $\mathcal{S}_{PT} \geq \epsilon \epsilon' I \gg 0$ , for some  $\epsilon' > 0$ .  $\square$

The proof of Theorem 6.2 was based on the following equivalence:

**Lemma 12.3 (ARE  $\Leftrightarrow$  WR IRE)** *Assume that  $\mathcal{D}$  is WR. Then the WR solutions of the ARE are exactly the solutions of the IRE for which  $\mathcal{F}$  is WR and  $F = 0$ .*

In the proof we also show that “and  $F = 0$ ” can be removed if, in (38),  $S$  is replaced by  $X^*SX$  and  $SK$  by  $X^*SK$ , where  $X = I - F$  (i.e.,  $\hat{\mathcal{X}}(s) = X - K_w(s - A)^{-1}B$ ); we call that variant the *extended ARE*. In Theorem 6.2 this corresponds to accepting WR  $J$ -optimal state-feedback pairs instead of merely WR  $J$ -optimal state-feedback operators.

**Proof of Lemma 12.3:** *Remark:* In this proof we also show that all solutions of the extended ARE are exactly all “WR” (meaning that  $H_B \subset \text{Dom}(K_w)$ ) solutions of the IRE except that for the solutions of this extended ARE we have to add the requirement  $H_B \subset \text{Dom}(K_w)$  (this requirement is redundant if  $S \in \mathcal{GB}(U)$ , because in 2.2° we show that  $H_B \subset \text{Dom}((B_w \mathcal{P})_w)$ , which implies that  $H_B \subset \text{Dom}(SK_w)$  (because  $H_B \subset \text{Dom}(C_w)$  because  $\mathcal{D}$  is WR)), and that we do not know whether  $\mathcal{H}$  and  $\mathcal{F}$  are well-posed (this is not a problem, since it is implicitly required by saying that  $(\mathcal{P}, S, [\mathcal{H} \mid \mathcal{F}])$  is a solution of the IRE or that  $(\mathcal{P}, S, K)$  is a WR solution of the ARE). This fact was originally shown in Proposition 9.8.10 of [M02], with an alternative, time-domain proof.

1°  $\widehat{\text{IRE}} \Rightarrow \text{ARE}$ : Multiply (47c) by  $(z - A)$  to the right and then let  $s \rightarrow +\infty$  to obtain (38c) on  $\text{Dom}(A)$  (note that  $\hat{\mathcal{X}}(+\infty) = I - \hat{\mathcal{F}}(+\infty) = 0$ , and that  $(s - A)^{-1}x_0 \rightarrow 0$  in  $\text{Dom}(A)$  for all  $x_0 \in H$ , by Lemma A.4.4(d3) of [M02]). Let first  $s \rightarrow +\infty$  and then  $z \rightarrow +\infty$  in (47b) to obtain (38b).

*Remark:* The limits of the  $B^*$ -terms below exist since so do the others.

2°  $\text{ARE} \Rightarrow \widehat{\text{IRE}}$ : Now  $\hat{\mathcal{X}}(s) = X - K_w V_s$ ,  $\hat{\mathcal{X}}(s)^* = X^* - V_s^* K$  ( $s \in \mathbb{C}_{\omega_A}^+$ ), by Lemma A.2(c) and regularity, where  $X := I$ ,  $V_s := (s - A)^{-1}B$ ,  $V_s^* := B_w^*(s - A)^{-*}$ . (This explicit  $X$  makes it easier to follow the computations and allows us to prove the more general result (“extended ARE” or eCARE) given in [M02] and mentioned below Lemma 12.3.) Naturally,  $\hat{\mathcal{D}}(s) = D + C_w V_s$ .

2.1° (47c): Multiply (47c) by  $z - A$  to the right to obtain

$$X^*SK - V_s^*K^*SK = -D^*JC - V_s^*(C^*JC + s^*\mathcal{P} + \mathcal{P}A). \quad (161)$$

Use (38a) to obtain  $X^*SK + D^*JC = -V_s^*(s^*\mathcal{P} - A^*\mathcal{P}) = -B_w^*\mathcal{P}$ , which is true, by (38c).

2.2° (47b): We have  $[I - r(r - A)^{-1}](z - A)^{-1} = A(r - A)^{-1}(z - A)^{-1} = (r - A)^{-1}[I - z(z - A)^{-1}]$ , hence  $[B_w^*\mathcal{P} - (B_w^*\mathcal{P})_w](z - A)^{-1}B = w\text{-}\lim_{r \rightarrow +\infty} B_w^*\mathcal{P}(r - A)^{-1}[I - z(z - A)^{-1}]B = w\text{-}\lim_{r \rightarrow +\infty} B_w^*\mathcal{P}(r - A)^{-1}B$ , which exists, by (38b), hence so does the weak limit  $(B_w^*\mathcal{P})_w(z - A)^{-1}B$ . By (38b) and the above,

$$B_w^*\mathcal{P}V_z - (B_w^*\mathcal{P})_wV_z = X^*SX - D^*JD, \quad (162)$$

where  $V_z := (z - A)^{-1}B$  (an alternative proof is given in Lemma 9.11.5(a) of [M02]). Apply (38c) to  $r(r - A)^{-1}x_0$  and let  $r \rightarrow +\infty$  to obtain

$$D^*JC_w x_0 + X^*SK_w x_0 = -(B_w^*\mathcal{P})_w x_0 \quad (163)$$

(in particular,  $(B_w^*\mathcal{P})_w x_0$  exists) for all  $x_0 \in \text{Dom}(C_w) \cap \text{Dom}(K_w)$ . Subtract the left side of (47b) from the right and use (163) and its dual to obtain

$$D^*JD - X^*SX - (B_w^*\mathcal{P})_wV_z - [(B_w^*\mathcal{P})_wV_s]^* + T \quad (164)$$

where  $T := V_s^*[(z + \bar{s})\mathcal{P} + C^*JC_w - K^*SK_w]V_z$ ,  $V_s^* := B_w^*(s - A)^{-*}$ . But  $[(z + \bar{s})\mathcal{P} + C^*JC_w - K^*SK_w] := w\text{-}\lim_{r \rightarrow +\infty} [(z + \bar{s})\mathcal{P} + C^*JC - K^*SK]r(r - A)^{-1}$ , and  $[\dots] = [(z + \bar{s})\mathcal{P} - A^*\mathcal{P} - \mathcal{P}A^*] = [(s - A)^*\mathcal{P} + \mathcal{P}(z - A)]$ , by (38a), hence

$$T = w\text{-}\lim_{r \rightarrow +\infty} [B_w^*\mathcal{P}r(r - A)^{-1}(z - A)^{-1}B + B_w^*(s - A)^{-*}\mathcal{P}r(r - A)^{-1}B] = (B_w^*\mathcal{P})_wV_z + (B_w^*\mathcal{P}V_s)^*. \quad (165)$$

Thus, (164) becomes  $D^*JD - X^*SX - [(B_w^*\mathcal{P})_wV_s]^* + (B_w^*\mathcal{P}V_s)^* = 0^* = 0$ , by (162).  $\square$

Above we also showed the following:

**Corollary 12.4** ( $\widehat{\text{ARE}} \Leftrightarrow \widehat{\text{IRE}}$ ) Any solution  $(\mathcal{P}, S, K)$  of the ARE having  $S \in \mathcal{GB}(U)$  satisfies the  $\widehat{\text{IRE}}$  (47). with  $\hat{\mathcal{X}}(s) := I - K_w(s - A)^{-1}B$ . If, in addition,  $S \gg 0$ , then  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{X} & \mathcal{F} \end{bmatrix}$  is a WR WPLS.  $\square$

(Use Lemma 12.6 for the last claim; weak regularity ( $H_B \subset \text{Dom}(K_w)$ ) was shown above.)

**Proof of Lemma 7.4:** Since  $B$  is bounded, now  $\mathcal{D}$  (and  $\mathcal{F}$  if any) is ULR and a control in WPLS form is necessarily given by a state-feedback pair, by Lemmata 6.3.16(b) and 8.3.18 of [M02]. In particular,  $\mathcal{S}^t$ -IRE or  $\hat{\mathcal{S}}$ -IRE implies the IRE and the  $\widehat{\text{IRE}}$ , by Lemma 10.5, hence the ARE, by Lemma 12.3. Conversely, if  $(\mathcal{P}, S, K)$  is a WR solution of the ARE, then it is admissible (because  $\mathcal{X}$  is ULR and  $\hat{\mathcal{X}}(+\infty) = 0$ ), hence we obtain the  $\widehat{\text{IRE}}$  (hence  $\mathcal{S}^t$ -IRE and  $\hat{\mathcal{S}}$ -IRE) from Lemma 12.3. If  $D^*JD \in \mathcal{GB}$ , then any solution of the ARE is WR, by Lemma 6.3.17 of [M02].  $\square$

**Proof of Theorem 8.1:** 0° We shall use the following assumptions, all of which will be established in the proof of Remark 8.4:

0.1°  $\mathcal{B} \subset \mathcal{A} \subset \text{TIC}$ , i.e.,  $\mathcal{B}(H_1, H_2) \subset \mathcal{A}(H_1, H_2) \subset \text{TIC}(H_1, H_2)$  for all Hilbert spaces  $H_1, H_2$ ,

0.2°  $\mathcal{A}$  is closed w.r.t. addition, composition, inversion, scalar multiplication and added stability ( $\mathbb{C}\mathcal{A}^{-1} + \mathcal{A}\mathcal{A} \subset \mathcal{A}$ , and  $e^{\omega\cdot}\mathcal{D}e^{-\omega\cdot} \in \mathcal{A}$  for all  $\omega < 0$ ,  $\mathcal{D} \in \mathcal{A}$ ).

(0.1° and 0.2° imply that  $\mathcal{B} \subset \mathcal{A}_\infty \subset \text{TIC}_\infty$  and that  $\mathbb{C}\mathcal{A}_\infty^{-1} + \mathcal{A}_\infty\mathcal{A}_\infty \subset \mathcal{A}_\infty$ , because the “ $\omega$ -shift” commutes with these operations.)

0.3°  $\mathcal{A}$  is closed w.r.t. spectral factorization, by Theorem 5.26.

0.4° The maps in  $\mathcal{A}$  are UR (uniformly regular), i.e., we have  $\|\hat{\mathcal{E}}(s) - \hat{\mathcal{E}}(+\infty)\| \rightarrow 0$ , as  $s \rightarrow +\infty$ , for all  $\mathcal{E} \in \mathcal{A}$  (hence for all  $\mathcal{E} \in \mathcal{A}_\infty$ ).

0.5° In (a2) we also use the following: 1.  $\mathcal{A} = \mathcal{A}^d$ . 2. If  $g_1, g_2 \in L^1(\mathbb{R}; \mathcal{B})$  and  $E \in \mathcal{B}$ , then  $\mathcal{E} := f * = g_1 * (E + g_2 *)$ , where  $f := g_1 E + g_1 * g_2 \in L^1(\mathbb{R}; \mathcal{B})$ , and  $\mathcal{E}_+ = f_+ * \in \mathcal{A}$ , where  $(\mathcal{E}_+ u)(t) := (\mathcal{E} \pi_{(-\infty, t)} u)(t)$ ,  $f_+ := \chi_{\mathbb{R}_+} f$ .

0.6° We have  $\|\mathcal{E}(ir) - \mathcal{E}(+\infty)\| \rightarrow 0$ , as  $|r| \rightarrow \infty$ , for all  $\mathcal{E} \in \mathcal{A}$ , by the Riemann–Lebesgue Lemma [M02].

(a1) By taking  $\alpha$  big enough, we have  $\mathcal{D}_+ \in \mathcal{A}$ , hence  $\mathcal{X}_+, \mathcal{X}_+^{-1} \in \mathcal{A}$  in the proof of Theorem 5.1 (by 0.3°). It follows that  $\mathcal{X}, \mathcal{M} \in \mathcal{A}_\infty$ . Therefore,  $\mathcal{F} = I - \mathcal{X}$ ,  $\mathcal{F}_\circ = \mathcal{M} - I$ ,  $\mathcal{N} = \mathcal{D}_\circ = \mathcal{D}\mathcal{M}$ ,  $\mathcal{B}_\circ\tau = \mathcal{B}\tau\mathcal{M}$  (see (25)) are in  $\mathcal{A}_\infty$ , by 0.2°.

By Theorem 7.2(iv), we have  $\mathcal{X}_+^* S \mathcal{X}_+ = \mathcal{D}_+^* J_+ \mathcal{D}_+$  with  $\mathcal{D}_+, \mathcal{X}_+ \in \mathcal{A}$ , hence  $\widehat{\mathcal{X}_+^* S \mathcal{X}_+} = \widehat{\mathcal{D}_+^* J_+ \mathcal{D}_+}$  on  $i\mathbb{R}$ , hence  $S = D_+^* J_+ D_+ = D^* J D$ , by 0.6°.

(a2) By (138), we have

$$\mathcal{K}_+^d \tau^t u = -\mathcal{C}_+^d J_+^* \mathcal{R} \mathcal{N}_+ S^{-1} \mathcal{R} \pi_+ \tau^t u = -\mathcal{C}_+^d \tau^t J_+ \mathcal{R} \mathcal{N}_+ S^{-1} \mathcal{R} \pi_{(-\infty, t)} u, \quad (166)$$

where  $(\mathcal{R}u)(t) := u(-t)$ . Set  $\mathcal{E}_1 := \mathcal{C}_+^d \tau$ . The top row of  $\mathcal{E}_1$  is in  $\mathcal{A}$ , by the assumption in (a2) (see (137)). One easily verifies that the bottom row of  $\mathcal{E}_1$  equals  $f \mapsto e^{-\alpha\cdot} \mathcal{A}^* * f$ , hence  $\mathcal{E}_1 \in \mathcal{A}$  (increase  $\alpha$  if necessary). Set  $\mathcal{E}_2 := -J_+ \mathcal{N}_+ S^{-1} \in \mathcal{A}$  to observe that  $(\mathcal{K}_+^d \tau u)(t) = (\mathcal{E}_1 \mathcal{R} \mathcal{E}_2 \mathcal{R} \pi_{(-\infty, t)} u)(t) \forall t \in \mathbb{R}$ , so that  $\mathcal{K}_+^d \tau \in \mathcal{A}$ , by 0.5° (because  $\mathcal{E}_2 = E + h * \Rightarrow \mathcal{R} \mathcal{E}_2 \mathcal{R} = E + h(-\cdot) *$ ). We conclude that  $\mathcal{K}_+^d \tau \in \mathcal{A}_\infty$  (since  $\mathcal{K} = e^{\alpha\cdot} \mathcal{K}_+$ , as noted below (138)). Since  $\mathcal{K}_\circ = \mathcal{M} \mathcal{K}$ ,  $\mathcal{C}_\circ = \mathcal{C} + \mathcal{N} \mathcal{K}_\circ$ , the remaining claims follow from this and (a1).

(a3) Apply Lemma 8.2 to  $\Sigma_\circ$  to get “ $\in \mathcal{A}_\omega$ ”. The latter claim follows from Theorem 4.7.

(b) Since  $\mathcal{X}$  is UR, we have  $X \in \mathcal{GB}(U)$  (Proposition 6.3.1(b1) of [M02]). Therefore, we can choose  $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$  so that  $F = 0$ , i.e., so that  $K$  is a UR  $J$ -optimal state-feedback operator, by (28). This leads to the ARE (38), by Theorem 6.2. Obviously, Theorem 6.2(i) implies Theorem 7.2(i), hence the equivalence holds. From 1° of the proof of Theorem 6.2 we observe that the limit converges in norm to  $S - D^* J D$ .

(c)  $\|\hat{f}(r + i\cdot)\|_\infty \leq \|e^{-r\cdot} f\|_1 \rightarrow 0$ , as  $r \rightarrow +\infty$ .  $\square$

Before proving the main result, Theorem 5.1, we explain how it was obtained. As mentioned above, Theorem 4.7 has already been known in the positive case. Our contribution was 1. to find the necessary and sufficient conditions in Theorem 7.2, particularly the “spectral factorization condition” (iv); 2. to show (Lemma 11.4(b)) that if the Popov

Toeplitz operator is uniformly positive ( $\mathcal{S}_{PT} \geq \epsilon I$ ), then so is the “shifted Popov function” ( $\hat{\mathcal{S}}(\alpha + i \cdot, \alpha + i \cdot) \geq \epsilon I$ ), so that the condition (iv) is satisfied by the standard positive spectral factorization result (Theorem 5.26(a)). See 2° below for details.

**Proof of Theorem 5.1:** 1° “If”: If  $[\mathcal{K} | \mathcal{F}]$  is  $J$ -optimal, then  $\mathcal{K}_\circ x_0 \in \mathcal{U}_*(x_0) \forall x_0$ .

2° “Only if”: Assume the FCC, so that the assumptions of Theorem 7.2 are satisfied, by Theorem 4.6. By Lemma 11.4(b), we have  $\hat{\mathcal{S}}(s, s) \geq \epsilon I$  on  $\mathbb{C}_{\omega_0}^+$ . Fix some  $\alpha > \omega_0$  to conclude that  $\widehat{\mathcal{D}}_+^*(ir)^* J_+ \widehat{\mathcal{D}}_+(ir) = \hat{\mathcal{S}}(\alpha + ir, \alpha + ir) \geq \epsilon I \forall r \in \mathbb{R}$ , i.e., that  $\mathcal{D}_+^* J_+ \mathcal{D}_+ \geq \epsilon I$ . Consequently, there is a spectral factorization  $\mathcal{D}_+^* J_+ \mathcal{D}_+ = \mathcal{X}_+^* S \mathcal{X}_+$  (i.e.,  $S \in \mathcal{GB}(U)$ ,  $\mathcal{X}_+ \in \mathcal{GTIC}(U)$ ), by Theorem 5.26. Thus, Theorem 7.2(iv)&(i) imply that there is a  $J$ -optimal state-feedback pair  $[\mathcal{K} | \mathcal{F}]$  for  $\Sigma$  over  $\mathcal{U}_*$ , with  $\mathcal{F} = I - \mathcal{K}$ .  $\square$

**Proof of Theorem 5.9:** 1° (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i): This is trivial.

2° (i) $\Rightarrow$ (iii): Assume (i). The map  $\tilde{\mathcal{D}} := \begin{bmatrix} \mathcal{D} \\ I \end{bmatrix}$  is  $I$ -coercive over  $\mathcal{U}_{\text{out}}^{\tilde{\Sigma}}$  (because  $\langle \tilde{\mathcal{D}}u, I \tilde{\mathcal{D}}u \rangle = \|\tilde{\mathcal{D}}u\|_2^2 = \|\mathcal{D}u\|_2^2 + \|u\|_2^2$ ; here  $\tilde{\Sigma} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ ,  $\mathcal{C} := \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix}$ ). Therefore, we can apply Theorem 5.1 to obtain an  $I$ -optimal (over  $\mathcal{U}_{\text{out}}^{\tilde{\Sigma}}$ ) state-feedback pair  $[\mathcal{K} | \mathcal{F}]$  for  $\tilde{\Sigma} := \begin{bmatrix} \Sigma \\ 0 & I \end{bmatrix}$ ; let  $\mathcal{P}$  be the corresponding solution of the IRE (see Theorem 10.1; then  $\mathcal{P} = \mathcal{C}_\circ^* I \mathcal{C}_\circ \geq 0$ ,  $S \gg 0$ ); let  $\tilde{\Sigma}_\circ$  be the corresponding closed-loop system of  $\tilde{\Sigma}$  and  $\Sigma_\circ$  that of  $\Sigma$  (Definition 3.5).

By Lemma 12.1, the maps  $\mathcal{C}_\circ$  and  $\tilde{\mathcal{D}}_\circ$  are stable. Since  $\tilde{\mathcal{D}}_\circ = \begin{bmatrix} \mathcal{D}_\circ \\ \mathcal{M} \end{bmatrix}$ , where  $\mathcal{N} := \mathcal{D}_\circ := \mathcal{D} \mathcal{M}$ ,  $\mathcal{M} := (I - \mathcal{F})^{-1}$ , the maps  $\mathcal{N}, \mathcal{M}$  are stable. Similarly,  $\mathcal{C}_\circ$  and  $\mathcal{K}_\circ$  are stable, hence  $\Sigma_\circ \in \text{SOS}$  (here  $\Sigma_\circ$  refers to  $\Sigma$  under  $[\mathcal{K} | \mathcal{F}]$ ). By, e.g., Lemma 10.4(c2), we have  $\tilde{\mathcal{D}}_\circ^* I \tilde{\mathcal{D}}_\circ = S$ . Let  $E := S^{1/2}$  and apply (28) to normalize  $S = \tilde{\mathcal{D}}_\circ^* \tilde{\mathcal{D}}_\circ = \mathcal{N}^* \mathcal{N} + \mathcal{M}^* \mathcal{M}$  to identity. By Lemma 12.2,  $\mathcal{N}, \mathcal{M}$  are q.r.c.  $\square$

**Proof of Corollary 5.16:** 1° We add a copy of  $u$  to the output, i.e., we define a WPLS  $\Sigma^e$  on  $(U \times W, H, U \times Y)$  by setting  $\mathcal{A}^e := \mathcal{A}$ ,  $\mathcal{B}^e := \begin{bmatrix} \mathcal{B} & \mathcal{H} \end{bmatrix}$ ,  $\mathcal{C}^e := \begin{bmatrix} 0 \\ \mathcal{C} \end{bmatrix}$ ,  $\mathcal{D}^e := \begin{bmatrix} I & 0 \\ \mathcal{D} & \mathcal{G} \end{bmatrix}$ . Set  $\mathcal{Q} := 0$ ,  $\mathcal{R} = \begin{bmatrix} 0 & I \end{bmatrix}$ ,  $\mathcal{Z}^s := \{0\}$ ,  $\mathcal{Z}^u := L^2$ ,  $\vartheta = 0$  to have

$$\mathcal{U}_*^{\Sigma^e}(x_0) = \{ \begin{bmatrix} u \\ 0 \end{bmatrix} \in L^2(\mathbb{R}_+; U \times W) \mid y := \mathcal{C}x_0 + \mathcal{D}u \in L^2 \} = \mathcal{U}_{\text{out}}(x_0) \times \{0\}. \quad (167)$$

With  $J^e := I \in \mathcal{B}(U \times Y)$  we get the cost function  $\mathcal{J}^e(x_0, \begin{bmatrix} u \\ 0 \end{bmatrix}) = \|y\|_2^2 + \|u\|_2^2$ , where  $u^e = \begin{bmatrix} u \\ w \end{bmatrix}$  is the input and  $y^e = \begin{bmatrix} y \\ u \end{bmatrix}$  the output of  $\Sigma^e$ . Since  $\|\begin{bmatrix} u \\ 0 \end{bmatrix}\|_{\mathcal{U}_*^{\Sigma^e}} = \max\{\|u\|_2, \|\begin{bmatrix} u \\ y \end{bmatrix}\|_2, \|0\|_{\mathcal{Z}^s}\}$ , we have  $\mathcal{S}_{PT}^e \gg 0$ , and  $\mathcal{P}^e$  and  $\mathcal{K}_{\circ 1}^e := \mathcal{K}_\circ^e \begin{bmatrix} I \\ 0 \end{bmatrix}$  are the same as  $\mathcal{P}$  and  $\mathcal{K}_\circ$  in the proof of Theorem 5.9, respectively, (by the uniqueness of the optimal control  $\mathcal{K}_{\circ 1}^e$ ), and  $\mathcal{K}_{\circ 2}^e = 0$  (since  $\mathcal{K}_\circ^e x_0 \in \mathcal{U}_*^{\Sigma^e}(x_0) \forall x_0$ ).

As in the last paragraph of the proof of Corollary 5.3, we see that  $\begin{bmatrix} 0 & I \end{bmatrix} \mathcal{M}^e \in \mathcal{B}(U \times W, W)$  and that we can have  $\widehat{\mathcal{M}}^e(\alpha) = \begin{bmatrix} \hat{\mathcal{M}}^{(\alpha)} & 0 \\ 0 & I \end{bmatrix}$ , so that  $\mathcal{M}_{11}^e = \mathcal{M}$  (being unique modulo constant, by (171), because  $\mathcal{B}_\circ^e \begin{bmatrix} I \\ 0 \end{bmatrix} = \mathcal{B}_\circ$  and  $\mathcal{K}_{\circ 1}^e = \mathcal{K}_\circ$ ), hence  $\begin{bmatrix} 0 & I \end{bmatrix} \mathcal{F}^e = \begin{bmatrix} 0 & I \end{bmatrix} (\mathcal{M}^e)^{-1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ ,  $\mathcal{F}_{11}^e = I - (\mathcal{M}_{11}^e)^{-1} = \mathcal{F}$ ,  $\mathcal{K}_1^e = (\mathcal{M}_{11}^e)^{-1} \mathcal{K}_{\circ 1}^e = \mathcal{M}^{-1} \mathcal{K}_\circ = \mathcal{K}$  and  $\mathcal{K}_2^e = 0$ , as required. By Lemma 12.1,  $\Sigma_\circ^e$  is SOS-stable, hence so is  $\tilde{\Sigma}_\circ$  (being a subset of  $\Sigma_\circ^e$ ).

2° *Q.r.c.*: The two first columns of the resulting closed-loop system  $\tilde{\Sigma}_\circ$  equal  $\Sigma_\circ$  extended by  $\begin{bmatrix} 0 & 0 \end{bmatrix}$ ; in particular,  $\mathcal{N} := \mathcal{D}_\circ = \tilde{\mathcal{D}}_\circ \begin{bmatrix} I \\ 0 \end{bmatrix}$  and  $\mathcal{M} = \hat{\mathcal{M}}_{11}$  are q.r.c. (by the choice of  $[\mathcal{K} | \mathcal{F}]$ ). But  $\text{TIC} \ni \tilde{\mathcal{M}} = (I - \mathcal{F})^{-1} = \begin{bmatrix} \mathcal{M} & \mathcal{M} \mathcal{E} \\ 0 & I \end{bmatrix}$  and  $\text{TIC} \ni \tilde{\mathcal{N}} = \begin{bmatrix} \mathcal{D} & \mathcal{G} \end{bmatrix} \tilde{\mathcal{M}} = \begin{bmatrix} \mathcal{N} & \mathcal{G}_\circ \end{bmatrix}$ , where  $\mathcal{G}_\circ = \mathcal{G} + \mathcal{N} \mathcal{E}$ . If  $\tilde{\mathcal{M}} \begin{bmatrix} u \\ w \end{bmatrix}, \tilde{\mathcal{N}} \begin{bmatrix} u \\ w \end{bmatrix} \in L^2$ , then  $w, \mathcal{M}(u + \mathcal{E}w) \in L^2$ , hence  $w, \mathcal{M}u \in L^2$  (since  $\mathcal{M} \mathcal{E} \in \text{TIC}$ ), and  $\mathcal{N}u + \mathcal{G}_\circ w \in L^2$ , hence  $\mathcal{N}u \in L^2$  (since  $\mathcal{G}_\circ \in \text{TIC}$ ), hence  $u \in L^2$  (since  $\mathcal{M}, \mathcal{N}$  are q.r.c.), hence  $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$  are q.r.c.  $\square$

**Proof of Theorem 5.17:** (By Theorem 5.9, these pairs  $[\mathcal{K} | \mathcal{F}]$  or  $\begin{bmatrix} \mathcal{K} \\ \mathcal{G} \end{bmatrix}$  (or any output-stabilizing pairs) exist iff  $\Sigma, \Sigma^d$  satisfy the output-FCC.)

1° Choose  $\mathcal{E}$  as in Corollary 5.16 to make  $(\Sigma_{\text{Joint}})_L$  SOS-stable (since it is contained in  $\tilde{\Sigma}_\circ$ ).

2°  $\tilde{\mathcal{N}}, \tilde{\mathcal{M}}$  are l.c. Since the maps in (35) are the inverses of each other, we observe that  $\begin{bmatrix} \tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix} \begin{bmatrix} -\mathcal{Y}_1 \\ \mathcal{X}_1 \end{bmatrix} = I$ .

3° “Moreover I/O-”: Actually, we have shown above that if  $[\mathcal{H}^d \mid \mathcal{G}^d]$  is any (by 1°) I/O-stabilizing (i.e., one that makes the I/O map  $(\mathcal{D}^d)_\circ$  of  $(\Sigma^d)_\circ$  stable) state-feedback pair for  $\Sigma^d$ , then  $\tilde{\mathcal{N}}, \tilde{\mathcal{M}}$  are l.c. By duality, the “moreover” claim holds.

4° D.c.f.;  $(\Sigma_{\text{Joint}})_L^d$  is SOS-stable: By 3°, we have the r.c.f. and l.c.f.  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1} = \tilde{\mathcal{M}}^{-1}\tilde{\mathcal{N}}$ . By Lemma 4.3(iii) of [S98a], we can find  $\tilde{\mathcal{X}}, \tilde{\mathcal{Y}} \in \text{TIC}$  that complete them (and  $\mathcal{X}, \mathcal{Y}$ ) to a d.c.f. (Given any  $\tilde{\mathcal{X}}_0, \tilde{\mathcal{Y}}_0 \in \text{TIC}$  for which  $\tilde{\mathcal{X}}_0\tilde{\mathcal{M}} - \tilde{\mathcal{Y}}_0\tilde{\mathcal{N}} = I$ , set  $\tilde{\mathcal{Y}} := \tilde{\mathcal{Y}}_0 + (\tilde{\mathcal{X}}_0\mathcal{Y}_1 - \tilde{\mathcal{Y}}_0\mathcal{X}_1)\tilde{\mathcal{M}}$ ,  $\tilde{\mathcal{X}} := \tilde{\mathcal{X}}_0 + (\tilde{\mathcal{X}}_0\mathcal{Y}_1 - \tilde{\mathcal{Y}}_0\mathcal{X}_1)\tilde{\mathcal{N}}$ .)

But the inverse of  $[\begin{smallmatrix} \mathcal{M} & \mathcal{Y} \\ \mathcal{N} & \mathcal{X} \end{smallmatrix}]$  in  $\text{TIC}_\infty(Y \times U)$  is given in (35) and it is unique, hence also the maps  $\mathcal{F}_L, \mathcal{E}_L$  must be stable ( $\in \text{TIC}$ ). We conclude that also  $(\Sigma_{\text{Joint}})_L^d$  is SOS-stable (its output map equals that of  $(\Sigma^d)_\circ$ ).

5° Externally stabilizing: We complete the proof by showing that any jointly admissible pairs  $[\mathcal{K} \mid \mathcal{F}], [\begin{smallmatrix} \mathcal{H} \\ \mathcal{G} \end{smallmatrix}]$  that make  $(\Sigma_{\text{Joint}})_L$  and  $(\Sigma_{\text{Joint}})_L^d$  SOS-stable actually make them externally stable (i.e., that also  $\mathcal{B}_L, \mathcal{H}_L, \mathcal{C}_L, \mathcal{K}_L$  are stable; it obviously also follows that  $\Sigma_\circ$  is externally stable). Now (cf. (6.170) and (6.171) of [M02])

$$\mathcal{B}_L = \mathcal{B}\mathcal{M} = \mathcal{B}_L\tilde{\mathcal{M}} - \mathcal{H}\tilde{\mathcal{M}}\mathcal{N} = \mathcal{B}_L\tilde{\mathcal{M}} - \mathcal{H}_L\mathcal{N} \quad (168)$$

is stable. Therefore,  $\mathcal{H}_L\tilde{\mathcal{M}} = \mathcal{H}_L + \mathcal{B}_L\mathcal{E}_L$  is stable, and so is  $\mathcal{H}_L\tilde{\mathcal{N}} = \mathcal{B}_L + \mathcal{B}(\mathcal{M}\mathcal{E}_L\tilde{\mathcal{N}} - I)$ , because  $\mathcal{B}(-\mathcal{Y}_1\tilde{\mathcal{N}} - I) = \mathcal{B}(-\mathcal{M}\tilde{\mathcal{X}}) = \mathcal{B}_L\tilde{\mathcal{X}}$ ; consequently,  $\mathcal{H}_L$  is stable (since  $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$  are l.c., by the d.c.f. (35)). By duality, also  $(\Sigma_{\text{Joint}})_L^d$  is externally stable.

6° “Moreover, SOS-”: By the above, also any other (by 1°) SOS-stabilizing  $[\mathcal{H}^d \mid \mathcal{G}^d]$  for  $\Sigma^d$  is externally stabilizing. By duality, any SOS-stabilizing  $[\mathcal{K} \mid \mathcal{F}]$  for  $\Sigma$  is externally stabilizing.

7° Final equivalence: We have shown above “(ii) $\Rightarrow$ (i)” (the converse is obvious). The proof of (i) $\Rightarrow$ (v) $\Rightarrow$ (ii) is obtained as in Theorem 7.2.4(c1) of [M02], using in place of Theorem 6.7.10(d)(viii) the fact that a WPLS is externally stable iff its I/O map is stable and the WPLS is input-detectable and output-stabilizable (since  $\mathcal{C} = \mathcal{C}_\circ - \mathcal{D}\mathcal{K}_\circ$  and similarly for  $\mathcal{B}$ ).  $\square$

**Proof of Lemma 6.5:** 1° The first UR claim is from Lemma 2.5 of [C03] (due to G. Weiss [WC99]).

2° (iv) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) $\Leftrightarrow$ (iii): By Theorems 10.1 and 6.2, conditions (i) and (iii), and (ii) and (iv) are equivalent. By definition, (ii) implies (i).

3° (i) $\Rightarrow$ (ii): Assume (i). By 1°, the I/O map  $[\begin{smallmatrix} \mathcal{D} \\ \mathcal{F} \end{smallmatrix}]$  of  $\Sigma_{\text{ext}}$  is UR. It follows that  $X := \hat{\mathcal{X}}(+\infty) = I - F$  is invertible, by Lemma 6.3.1(b1) of [M02], hence  $F$  can be normalized to zero, by (28), hence (ii) holds.

4° w-lim = 0: Let  $s = z \rightarrow +\infty$  in (47b) to obtain that  $S = D^*JD$  (since  $2s\|\mathcal{P}\|Ms^{-1-2\epsilon} \rightarrow 0$ ).

5° Positively  $J$ -coercive case: This follows from Theorem 5.1.

6° Bounded  $B$ : Naturally, (v) is necessary. Conversely, (v) leads to (33), and the generator  $K_{\text{opt}}$  of  $\mathcal{K}_{\text{opt}}$  is a uniformly line-regular state-feedback operator for  $\Sigma$  (see Lemma 8.3.18 of [M02] for details). Since obviously  $\Sigma_\circ[\begin{smallmatrix} I \\ 0 \end{smallmatrix}] = \Sigma_{\text{opt}}$ , the operator  $K_{\text{opt}}$  is  $J$ -optimal. By continuity, w-lim = 0.  $\square$

We obtain the  $\widehat{\text{IRE}}$  once the  $\hat{\mathcal{S}}$ -IRE holds at a single point (when we use characteristic functions in place of transfer functions and do not consider well-posedness):

**Lemma 12.5 ( $\hat{\mathcal{S}}$ -IRE $\Rightarrow$  $\widehat{\text{IRE}}$ )** Assume that the  $\hat{\mathcal{S}}$ -IRE (44) holds (with  $\hat{\mathcal{D}}$  in place of  $\hat{\mathcal{D}}$ ) for some  $s, z \in \rho(A)$ ,  $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ ,  $\widehat{\mathcal{K}_{\text{opt}}}(z) \in \mathcal{B}(H, U)$ .

Fix this  $z$ . Define  $\tilde{\mathcal{X}}(s) := I - (z - s)K(s - A)^{-1}(z - A)^{-1}B \in \mathcal{B}(U) \ \forall s \in \rho(A)$ ,  $S := \hat{\mathcal{S}}(z, z)$ ,  $K := \widehat{\mathcal{K}_{\text{opt}}}(z)(z - A) \in \mathcal{B}(\text{Dom}(A), U)$  to obtain the  $\widehat{\text{IRE}}$  (47) for  $s = z$  (replace  $\hat{\mathcal{D}}$  by  $\hat{\mathcal{D}}$  and  $\hat{\mathcal{X}}$  by  $\tilde{\mathcal{X}}$ ). By Lemma 9.5(e) and the proof of Lemma 10.2, it follows that (47) holds for all  $s, z \in \rho(A)$ .  $\square$

In suitably positive problems, such as the LQR problem or most other problems of Section 5, we typically have  $S \gg 0$ ,  $\mathcal{P} \geq 0$ . In this case the maps in Lemma 12.5 are well-posed:

**Lemma 12.6** ( $\widehat{\text{IRE}} \& S \gg 0 \Rightarrow \text{SOS}$ ) Assume that  $\mathcal{P} \geq 0$ ,  $S \gg 0$  and  $K \in \mathcal{B}(\text{Dom}(A), U)$  are s.t. (47) (the  $\widehat{\text{IRE}}$ ) holds (use  $\check{\mathcal{D}}$  in place of  $\hat{\mathcal{D}}$ ) for some  $s = z \in \rho(A)$  and some  $\hat{\mathcal{X}}(z) \in \mathcal{B}(U)$ .

(a) Then  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$  is a solution of the IRE and the  $\widehat{\text{IRE}}$  and  $[\frac{\mathcal{A}}{\mathcal{K}} \mid \frac{\mathcal{B}}{\mathcal{F}}]$  is a WPLS, where  $\hat{\mathcal{X}}(s) := K(s - A)^{-1}$  and  $\hat{\mathcal{X}}(s) := \hat{\mathcal{X}}(z) + (s - z)K(z - A)^{-1}(s - A)^{-1}B$  (for the fixed  $z$  and  $\hat{\mathcal{X}}(z)$  of the previous paragraph),  $\mathcal{F} := I - \mathcal{X}$ .

(b) Assume, in addition, that  $\mathcal{C} = [\mathcal{C}_1]$ ,  $\mathcal{D} = [\mathcal{D}_1]$ ,  $J = [\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}] \gg 0$  for some operators  $\mathcal{C}_1, \mathcal{D}_1$  (then the above assumption  $S \gg 0$  becomes redundant). If 1.  $\hat{\mathcal{X}}(s) \in \mathcal{GB}(U)$  for some  $s \in \rho_\infty(A)$ , 2.  $\dim U < \infty$ , 3.  $B$  is not maximally unbounded (or  $\mathcal{X}$  is UR) and  $X \in \mathcal{GB}(U)$ , or 4.  $[\mathcal{K} \mid \mathcal{F}]$  is admissible, then  $[\mathcal{K} \mid \mathcal{F}]$  is SOS-stabilizing.

See Proposition 2.2.5 of [M02] for further sufficient conditions for the last claim. Note from Lemma A.2 that if  $[\frac{\mathcal{A}}{\mathcal{K}} \mid \frac{\mathcal{B}}{\mathcal{F}}]$  is a WPLS, then the formulas for  $\hat{\mathcal{X}}$  and  $\hat{\mathcal{X}} := I - \hat{\mathcal{F}}$  are as in Lemma 12.6.

The last paragraph of Lemma 12.6 together with Lemma 12.5 shows that if the LQR- $\mathcal{S}$ -IRE has a nonnegative solution at a single point  $z = s \in \rho_\infty(A)$ , then this solution is SOS-stabilizing; in particular, then the output-FCC holds and there is a smallest nonnegative solution (see Corollary 7.5(c)).

**Proof of Lemma 12.6:** (See Sections 9.12 and 10.7 of [M02] for similar results.)

1°  $\mathcal{K}$ : By (the proof of) Lemma 9.2, we observe that (46a) holds “on  $\text{Dom}(A) \times \text{Dom}(A)$ ”. Since  $S \gg 0$ , it follows from (46a) that  $K\mathcal{A}^t : \text{Dom}(A) \rightarrow L^2([0, t]; U)$  extends continuously to  $\mathcal{K}^t : H \rightarrow L^2([0, t]; U)$ . Obviously,  $[\frac{\mathcal{A}}{\mathcal{K}}]$  is (the left column of) a WPLS, hence (46a) holds.

2°  $\mathcal{X}$ : As in Lemma 12.5, we observe that the  $\widehat{\text{IRE}}$  (47) holds for all  $s, z \in \rho(A)$ . Fix some  $\omega > \max\{\omega_A, 0\}$ . Then the right-hand-side of (47b) is bounded on  $\mathbb{C}_\omega^+$  (since  $2 \text{Re } s \|(s - A)^{-1}B\|^2 \leq \|\mathcal{B}\|_{\mathcal{B}(L_\omega^2, H)}^2$  for  $s \in \mathbb{C}_\omega^+$ , by, e.g., (b3) on p. 176 of [M02]), hence so is  $\hat{\mathcal{X}}$ . By Lemma 6.3.15 of [M02], it follows that  $[\frac{A}{-K} \mid \frac{B}{\mathcal{X}}]$  are the generators of a WPLS  $[\frac{\mathcal{A}}{-\mathcal{K}} \mid \frac{\mathcal{B}}{\mathcal{X}}]$  (where the value of  $\hat{\mathcal{X}}$  could be fixed arbitrarily at a single point had we not already done it). The IRE follows from Lemma 10.2.

(b) 1° We first show that any of 2., 3. and 4. implies 1.: 4. If  $[\mathcal{K} \mid \mathcal{F}]$  is admissible (i.e.,  $\mathcal{X} \in \mathcal{GTIC}_\infty(U)$ ), then  $\hat{\mathcal{X}}(s) \in \mathcal{GB}(U)$  for each  $s$  in some right half-plane. 2. From (47b) we observe that  $\hat{\mathcal{X}}(s)^* S \hat{\mathcal{X}}(s) \geq J_{22} \gg 0 \forall s \in \rho(A)$ ; this shows the invertibility of  $\hat{\mathcal{X}}(s)$  for all  $s$  if  $\dim U < \infty$ . 3. If  $B$  is not maximally unbounded, then  $\hat{\mathcal{X}}$  is UR and hence  $X := \hat{\mathcal{X}}(+\infty) \in \mathcal{GB}(U)$  implies that  $\hat{\mathcal{X}}(s) \in \mathcal{GB}(U)$  for real  $s$  big enough.

Thus, we may assume that  $\hat{\mathcal{X}}(s_0)$  is invertible for some  $s_0 \in \rho_\infty(A)$ ; but this leads to  $S \geq \hat{\mathcal{X}}(s_0)^{-*} J_{22} \hat{\mathcal{X}}(s_0)^{-1} \gg 0$ , so the assumption  $S \gg 0$  is now redundant.

2°  $\hat{\mathcal{X}} \in \mathcal{GH}_\infty$ , i.e.,  $[\mathcal{K} \mid \mathcal{F}]$  is admissible: From  $\hat{\mathcal{X}}(s)^* S \hat{\mathcal{X}}(s) \geq J_{22} \gg 0$  we deduce that  $\hat{\mathcal{X}}(s)^{-1}$  is uniformly bounded (wherever it exists). Since  $\rho_\infty(A)$  is connected, it follows that  $\hat{\mathcal{X}}(s)^{-1}$  exists for all  $s \in \rho_\infty(A)$ .

3°  $\Sigma_\circ$  is SOS-stable: From (109) we observe that  $\int_0^t \|(\mathcal{C}_\circ x_0)(t)\|^2 dt \leq \langle x_0, \mathcal{P} x_0 \rangle \forall x_0 \in H$ , hence  $\|\mathcal{C}_\circ\|_{\mathcal{B}(H, L^2)}^2 \leq \|\mathcal{P}\| < \infty$ . But  $\mathcal{C}_\circ = \mathcal{C} + \mathcal{D}\mathcal{K}_\circ = [\begin{smallmatrix} \mathcal{C}_1 + \mathcal{D}_1 \mathcal{K}_\circ \\ \mathcal{K}_\circ \end{smallmatrix}]$ , hence also  $\mathcal{K}_\circ$  is stable. From (111) we observe that  $\mathcal{N}^t$  is uniformly bounded, hence  $\mathcal{N} \in \text{TIC}$ . But  $\mathcal{N} := \mathcal{D}\mathcal{M} = [\begin{smallmatrix} \mathcal{D}_1 \mathcal{M} \end{smallmatrix}]$ , hence  $\mathcal{M} \in \text{TIC}$  too.  $\square$

**Proof of Corollary 7.5:** Claims (a) and (b) were established on p. 37. Most of claim (c) follows from Sections 10.7 and 10.1 of [M02], but we give here a self-contained proof.

Since a  $\mathcal{U}_*$ -stabilizing solution is admissible (and  $\mathcal{P} \geq 0$  since the cost function  $\mathcal{J}$  is nonnegative), the necessity follows from (a) or (b). Below we establish the sufficiency and further claims.

Let  $\tilde{\Sigma}, \tilde{J}$  denote the system and cost operator whose IRE is used in the result under study (so  $\tilde{\Sigma} := [\begin{smallmatrix} \Sigma \\ 0 \end{smallmatrix} \mid I]$  and  $\tilde{J} = I$  in the proof of Theorem 5.9, p. 62; we need do not study its special case, Corollary 5.10).

1° Assume that  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$  is a solution of the IRE for  $\tilde{\Sigma}, I$  with  $\mathcal{P} \geq 0$  and  $[\mathcal{K} \mid \mathcal{F}]$  admissible for  $\tilde{\Sigma}$  (equivalently, to  $\Sigma$  or to any other extension of  $[\mathcal{A} \mid \mathcal{B}]$ ) or  $\dim U < \infty$ : By Lemma 12.6,  $S \gg 0$  and  $[\mathcal{K} \mid \mathcal{F}]$  is SOS-stabilizing. By  $\tilde{\Sigma}_\circ$  and  $\Sigma_\circ$  we denote the closed-loop systems corresponding to  $\tilde{\Sigma}$  and  $\Sigma$  (under  $[\mathcal{K} \mid \mathcal{F}]$ ).



1.1° For the output IRE (i.e., the one in for Theorem 5.9) we conclude that also  $\Sigma_\odot$  is then SOS-stable (being contained in  $\tilde{\Sigma}_\odot$ ), so the output-FCC holds for  $\Sigma$ . Thus, we have the sufficiency.

If  $\mathcal{P}$  is the  $\mathcal{U}_{\text{out}}^{\tilde{\Sigma}}$ -stabilizing one and  $\mathcal{P}'$  (with some  $S', [\mathcal{K}' \mid \mathcal{F}']$ ) is any other admissible nonnegative solution, then  $\mathcal{K}'_\odot x_0 \in \mathcal{U}_{\text{out}}^{\tilde{\Sigma}}(x_0) \forall x_0$  (being output stabilizing, as noted above), and  $\mathcal{P}' \geq \tilde{\mathcal{C}}_\odot^* J \tilde{\mathcal{C}}_\odot$ , by (109), hence, for  $u' := \mathcal{K}'_\odot x_0$ ,  $\tilde{y}' := \tilde{\mathcal{C}}_\odot x_0 + \tilde{\mathcal{D}} u' = \tilde{\mathcal{C}}_\odot' x_0$ ,  $u := \mathcal{K}_\odot x_0$ , we have  $\langle x_0, \mathcal{P}' x_0 \rangle \geq \langle y', J y' \rangle = \mathcal{J}(x_0, u') \geq \mathcal{J}(x_0, u) = \langle x_0, \mathcal{P} x_0 \rangle$ . Thus,  $\mathcal{P}$  is the smallest admissible nonnegative solution.

1.2° For the state IRE (Corollary 5.2), we have  $\tilde{\mathcal{C}} = \begin{bmatrix} \mathcal{A} \\ 0 \end{bmatrix}$ ,  $\tilde{\mathcal{D}} = \begin{bmatrix} \mathcal{B}^\tau \\ I \end{bmatrix}$ , hence

$$\tilde{\mathcal{C}}_\odot := \tilde{\mathcal{C}} + \tilde{\mathcal{D}} \mathcal{K}_\odot = \begin{bmatrix} \mathcal{A} + \mathcal{B}^\tau \mathcal{K}_\odot \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_\odot \\ 0 \end{bmatrix}, \quad (169)$$

hence  $\mathcal{A}_\odot x_0 \in L^2 \forall x_0 \in H$ , hence  $\Sigma_\odot$  is exponentially stable, by Lemma 2.2, hence  $(\mathcal{P}, S, [\mathcal{K} \mid \mathcal{F}])$  is  $\mathcal{U}_{\text{exp}}$ -stabilizing for  $\Sigma$  (hence unique) and the state-FCC holds for  $\Sigma$ .

2° The ARE: By Corollary 12.4, any nonnegative solution of the ARE solves the  $\widehat{\text{IRE}}$  (47) and makes  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{K} & \mathcal{F} \end{bmatrix}$  a WR WPLS with  $X = I$  (i.e.,  $\hat{\mathcal{F}}(+\infty) = I - X = 0$ ). Thus, if  $\dim U < \infty$  or  $B$  is not maximally unbounded or  $\mathcal{X}$  is UR, then  $[\mathcal{K} \mid \mathcal{F}]$  is SOS-stabilizing (hence admissible), by Lemma 12.6, so the sufficiency follows from 1°.  $\square$

**Proof of Corollary 6.6:** Set  $\tilde{\mathcal{C}} := \begin{bmatrix} \mathcal{C} \\ \mathcal{A} \\ 0 \end{bmatrix}$ ,  $\tilde{\mathcal{D}} := \begin{bmatrix} \mathcal{D} \\ \mathcal{B}^\tau \\ I \end{bmatrix}$  to make the output of  $\tilde{\Sigma} := \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  equal to  $(y; x; u)$ , where  $y$  is the output of  $\Sigma$ , both under the initial state  $x_0$  and input  $u$ . Obviously, the equation (41) equals (38) for  $\tilde{\Sigma}$  and  $J := \text{diag}(Q, T, R)$ ;  $\tilde{\Sigma}$  is positively  $J$ -coercive; and  $\tilde{\Sigma}$  is exponentially detectable (by Lemma 6.6.25 of [M02]), hence estimatable. Moreover, the FCC condition obviously equals the FCC for  $\tilde{\Sigma}$  and  $\mathcal{U}_{\text{out}}$ . By Lemma 6.5, conditions (i)–(v) are equivalent, hence the FCC holds iff (41) has a  $\mathcal{U}_{\text{out}}$ -stabilizing solution. The rest follows from Corollary 7.5(c) (for Theorem 5.9) applied to  $\begin{bmatrix} Q^{1/2} \mathcal{C} \\ T^{1/2} \mathcal{A} \end{bmatrix}$ ,  $\begin{bmatrix} Q^{1/2} \mathcal{D} E \\ T^{1/2} \mathcal{B}^\tau E \end{bmatrix}$  and  $R^{1/2} u$  in place of  $\mathcal{C}$ ,  $\mathcal{D}$  and  $u$ , respectively, where  $E := R^{-1/2} u$  (alternatively, slightly modify its proof (for our different  $J$ )). By Theorem 5.9,  $K$  is (q.r.c.)-SOS-stabilizing. In (a), also  $\mathcal{A}_\odot x_0 \in L^2 \forall x_0$ , hence then  $K$  is exponentially stabilizing.  $\square$

**Proof of Theorem 7.6:** We actually show the claim for  $\mathcal{S}^t$ -IRE's (equivalently,  $\Sigma_{\text{opt}}$ -IRE's), to obtain a more general claim. Let  $\mathcal{K}_\odot$  and  $\tilde{\mathcal{K}}_\odot$  be controls in WPLS form for  $\Sigma$  and let  $(\mathcal{P}, \mathcal{K}_\odot), (\tilde{\mathcal{P}}, \tilde{\mathcal{K}}_\odot)$  satisfy the  $\mathcal{S}^t$ -IRE. Assume that  $\mathcal{A}_\odot^t x_0 \rightarrow 0$  as  $t \rightarrow +\infty$ .

1° Compute  $(\tilde{\mathcal{K}}_\odot^*)^*(52)+(53)$  to obtain  $\mathcal{P} = (\mathcal{A}_\odot^t)^* \mathcal{P} \mathcal{A}_\odot^t + (\tilde{\mathcal{C}}_\odot^t)^* J \tilde{\mathcal{C}}_\odot$ . Exchange  $\mathcal{P}, \mathcal{K}_\odot$  and  $\tilde{\mathcal{P}}, \tilde{\mathcal{K}}_\odot$  to conclude that  $\mathcal{P} - \tilde{\mathcal{P}}^* = (\mathcal{A}_\odot^t)^* (\mathcal{P} - \tilde{\mathcal{P}}) \mathcal{A}_\odot^t$ , which converges to zero weakly, as  $t \rightarrow +\infty$ , if also  $\mathcal{A}_\odot^t \rightarrow 0$  strongly. Thus,  $\mathcal{P}$  is the unique strongly internally stabilizing solution of the  $\mathcal{S}^t$ -IRE.

2° Assume that  $\mathcal{S} \geq 0$ . Given any  $x_0 \in H$ , set  $u := \mathcal{K}_\odot x_0$ ,  $\tilde{u} := (\mathcal{K}_\odot - \tilde{\mathcal{K}}_\odot) x_0 \in L_{\text{loc}}^2(\mathbb{R}_+; U)$ ,  $y := \mathcal{C} x_0 + \mathcal{D} u = \tilde{\mathcal{C}}_\odot x_0 + \mathcal{D} \tilde{u}$ . Now  $x_T := x(T) = \mathcal{A}^T x_0 + \mathcal{B}^T u = \mathcal{A}_\odot^T x_0 + \mathcal{B}^T \tilde{u}$ , hence

$$\langle y, \pi_{[0,T)} J y \rangle = \langle x_0, \tilde{\mathcal{P}} x_0 \rangle - \langle x_T, \tilde{\mathcal{P}} x_T \rangle + \langle \tilde{u}, \mathcal{S}^t \tilde{u} \rangle, \quad (170)$$

by (54), (52) and (43b) (with tildes). But  $\langle y, \pi_{[0,T)} J y \rangle \rightarrow \langle x_0, \mathcal{P} x_0 \rangle$  (by (54)) and  $x_T \rightarrow 0$ , as  $T \rightarrow +\infty$ , hence  $\mathcal{S}^t \geq 0$  implies that  $\langle x_0, \mathcal{P} x_0 \rangle \geq \langle x_0, \tilde{\mathcal{P}} x_0 \rangle$ .  $\square$

**Proof of Theorem 11.2:** (a) This follows from Proposition 10.3.2(e2) (ULR from Theorem 9.2.3) of [M02].

(b) This follows from 10.3.2(e1)&(e2) and 9.2.2(1.)&(3.)&(4.) of [M02].

(c) 1° Case  $\mathcal{A}B \in L_{\text{loc}}^1$ : 10.3.2(e2). 2° Case  $B$  not maximally unbounded: By Corollary 7.5(b) and Theorem 5.1, (i) implies that the ARE has a  $\mathcal{U}_{\text{exp}}$ -stabilizing solution with  $S = D^* J D \gg 0$  (in fact, also this is an equivalent condition, by Proposition 9.9.12 of [M02]), hence  $D^* D \gg 0$ , hence (ii) holds. The rest follows from 10.3.2(e1)&(e2) of [M02].  $\square$

**Proof of Corollary 8.3:** (a) Combine (the proof of) Corollary 7.5(a) with Theorem 8.1(a1).

(b) This follows from (a) (because  $\mathbb{C}\mathcal{A}_\infty^{-1} + \mathcal{A}_\infty\mathcal{A}_\infty \subset \mathcal{A}_\infty$ , as noted in the proof of Theorem 8.1), as one observes from the proofs of the corollaries and the remark (see formulas (6.170) and (6.171) on p. 241 of [M02] for the claims on  $L$  and  $\tilde{L}$ ).

(c) The first claim follows from Corollary 7.5(a) and Theorem 8.1(c). The other two claims can be proved as the original ones (the only exception is that if  $B$  is maximally unbounded, we obtain the uniform regularity of  $\mathcal{F}$  in the same way as the weak regularity was obtained in the proof of Corollary 7.5 (now  $Ks(s - A)^{-1}x_0$  converges uniformly  $\forall x_0 \in H_B$ ; a more detailed proof is given in Lemma 9.11.5(e) of [M02]), because we have here required the “w-lim” in the ARE[s] to converge uniformly;

(d) This follows from Theorem 8.1(a3).  $\square$

**Proof of Remark 8.4:** Practically the same proofs still hold. We give below some details.

1° *Case*  $\mathcal{A} = \text{MTIC}^{\text{L}^1, \mathcal{B}\mathcal{C}}$ : Observe first that if  $T$  is a compact operator, then so are  $T^*$ ,  $ST$ ,  $TS$  and  $(X + T)^{-1} - X^{-1}$  (whenever they exist). By Theorem 8.1, it obviously suffices to prove that  $f(t)$  is compact for a.e.  $t$ , equivalently, that  $\hat{f}(z)$  is compact for all  $z$  on some right half-plane, where  $\mathcal{X} = X + f^*$  (since then the same holds for  $\mathcal{M} = \mathcal{X}^{-1}$  etc.), and that  $(z - A)^{-*}K^*$  is compact for all  $z$  on some right half-plane if  $(z - A)^{-*}C^*$  is. Multiply the  $\widehat{\text{IRE}}$  (47) by  $S^{-1}\hat{\mathcal{X}}(s)^{-*}$  to the left to observe this. For “finite-dimensional”, the same proof applies, mutatis mutandis.

2° *Cases*  $\mathcal{A} = \mathcal{A}_{H^2}$  and  $\mathcal{A} = \mathcal{A}_2$ : The first claims follow easily from Theorem 8.4.9 of [M02]. The equivalence with Theorem 6.7(3.) is from Lemma 6.8.1(a)&(d1) of [M02].  $\square$

**Notes for Section 12:** Lemma 12.2 seems to be new, whereas Lemma 12.1 (from [M02]) is a simple generalization of a classical result.

## 13 Conclusions

We summarize here the Riccati equation and optimization theory developed in this article, thus explaining how and to which extent the finite-dimensional results can be extended to WPLSs. The general setting being thus resolved, it seems that in the future the WPLS RE research should focus on the special cases where these results can be strengthened to give better applicability and on nonstandard REs (cf. [C03] and [M03b]).

For finite-dimensional  $U, H, Y$ , the equivalence of the following conditions is fairly well known:

- (i) **( $\exists! u_{\text{opt}}$ )** For each initial state  $x_0$  there is a unique optimal control.
- (ii) **( $u(t) = Kx(t)$ )** There is a unique optimal state-feedback operator.
- (iii) **(FCC & coercive)** The FCC holds and the cost function is  $J$ -coercive.
- (iv) **(RE)** The Riccati equation (ARE) has a stabilizing solution.

The cost functions (2) and (8) are  $J$ -coercive, and so is any other cost function that dominates the natural square norm of the input (p. 15) (otherwise the “infimal cost” would be achieved by no input or by many inputs). The FCC means that there are some admissible inputs for each initial state  $x_0$ .

Also in the infinite-dimensional case it has been known that roughly the same four conditions are equivalent even when  $A$  and  $C$  are unbounded operators (if  $B$  is bounded).

In this article, we have generalized this equivalence to the class of WPLSs, thus allowing for rather unbounded  $A, B, C$ . Our main results consist of the results “1.–3c.” below on the equivalence of (i)–(iv), and on the corollaries of them (particularly of “3b.”, including rather indirect ones, such as the results of Section 5):

1. If the system is **sufficiently regular**, then (i)–(iv) are equivalent.<sup>8</sup>
2. If the system is **weakly regular**, then (ii) and (iv) are equivalent (Theorem 6.2).
- 3a. If we replace the (infinitesimal algebraic) RE by the integral RE (*IRE*), then (ii) and (iv) are equivalent for **any WPLS** (Theorems 10.1 and 7.2).
- 3b. In fact, for the IRE, (i)–(iv) are equivalent if the cost is nonnegative (Theorem 5.1 with 3a.&3c.).
- 3c. For its variant, the  $\mathcal{S}^t$ -IRE, (i)–(iv) are equivalent in general if we allow for possibly ill-posed state feedback in (ii) (Theorem 7.1).

Examples of “1.” are (roughly) Theorems 6.1, 11.2(vii) and Corollary 6.6; Remark 9.9.14 of [M02]; for  $J$ -coercive systems also Theorem 8.1(b) and Remark 8.4; for positively  $J$ -coercive systems also Theorem 6.5, Corollaries 7.5(b)&(c) and 8.3(a)&(c). Except for Theorem 6.1, these results are new (except that 11.2 and 8.4 are from [M02], whose Chapters 9–10 also contain further results). In “3a.” and “3b.”, (ii) refers to a unique pair (modulo (28)), not necessarily to an operator.

We have already presented 2. in [M97] in the stable case and 1.–3a. in [M02] in the general case. In this article we have repeated most of them and established 3b. and 3c. Most earlier results were special cases of “1.”; e.g., in [vK93] at least most implications can be found, for smooth Pritchard–Salamon systems.

In the WPLS setting, the stable case of the implication (ii) $\Rightarrow$ (iv) (and (iii) $\Rightarrow$ (ii) for the Wiener class) was originally solved in [S97] and [WW97]. The implication (iii) $\Rightarrow$ (i) was established in [FLT88] (for WPLSs having bounded  $C$ ; see [Z96] for general WPLSs) in the case of the standard LQR cost function  $\|y\|_2^2 + \|u\|_2^2$ .

As is well-known, in some cases a fifth equivalent condition is the existence of a (coprime)  $J$ -inner factorization of the I/O map (a spectral factorization in the stable case); see [M02] for details (see Theorem 5.9(iii) and Lemma 10.7 for a special case).

It has been known for the standard LQR cost function that the optimal state is generated by a  $C_0$ -semigroup and that the optimal control (and output) is generated by an admissible output operator for this (closed-loop) semigroup, as in [FLT88] and [Z96]. It has not been known that the output is admissible also for the original semigroup, or that the state-feedback loop is well-posed w.r.t. external perturbation (i.e., that  $\Sigma_{\text{ext}}$  and  $\Sigma_\odot$  are WPLSs; see pp. 16&11). These facts are contained in “3b.” and they do not hold for indefinite ( $\mathcal{J}(0, \cdot) \not\geq 0$ ) cost functions, by Example 8.4.13 of [M02].

Nevertheless, even in the indefinite case (3c.), we have the “ARE on  $\text{Dom}(A + BK)$ ” (104) whenever  $\mathcal{D}$  is uniformly regular and a unique  $J$ -optimal control exists for each initial state. The solution of this ARE leads to the optimal control  $u(t) = -(D^*JD)^{-1}(B_w^*\mathcal{P} + D^*JC_w)x(t)$  for a.e.  $t > 0$ . To get AREs given on  $\text{Dom}(A)$  (such as (38)), one has to restrict to “2.”, and usually one wants to use further assumptions to simplify the ARE (as in “1.”).

There is some ongoing research on the computational aspects of the ARE, but further results are needed for sufficient applicability. Thus, the main contribution of this article consists of the abstract Riccati equation and optimization theory and of the stabilization and factorization results of Section 5.

## A Symbols $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{\mathcal{D}}_\Sigma, \dots$

In this appendix we present the frequency-domain symbols of WPLSs, and recall that  $\begin{bmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}}\tau \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{bmatrix} : \begin{bmatrix} x_0 \\ \hat{u} \end{bmatrix} \rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$  holds on  $\mathbb{C}_\omega^+$ , when  $\Sigma$  is  $\omega$ -stable,  $u \in L_\omega^2(\mathbb{R}_+; U)$  and  $x_0 \in H$ . Here  $\hat{\mathcal{A}}(s) = (s - A)^{-1}$ ,  $\hat{\mathcal{B}}\tau = (s - A)^{-1}B$ ,  $\hat{\mathcal{C}} = C(s - A)^{-1}$  (and  $\hat{\mathcal{D}} = D + C_w(s - A)^{-1}B$  if  $\hat{\mathcal{D}}$  is weakly regular), and  $\hat{u}, \hat{x}, \hat{y}$  are the Laplace transforms of  $u, x, y$ .

We also record some corollaries on “compatible pairs”, to be referred in this article in the regular case only and in [M03b] in the general case.

<sup>8</sup>To be exact, claims 1.–3c. are true when  $\dim U < \infty$  and  $\mathcal{U}_* = \mathcal{U}_{\text{exp}}$  or when we assume the cost to be coercive (as usual); otherwise (iii) is not implied by the other conditions. Moreover, in (iv) we have required the indicator (the “ $S$ ” or  $\mathcal{S}^t$  on pp. 30, 34, 36) to be one-to-one, although (ii) and (iv) are equivalent even without that assumption if the word “unique” is deleted (pp. 49&30; of course, nonuniqueness can only happen when the cost function is noncoercive (singular)).

Not all WPLSs are weakly regular (if they are, the values  $C_c = C_w$ ,  $D_c = D$  will do below), but yet (1), (5)–(7) and other classical equations can be recovered for all WPLSs:

**Lemma A.1 (Compatible pair  $(C_c, D_c)$ )** *Let  $\Sigma$  be a WPLS on  $(U, H, Y)$ . Then there are a Banach space  $W$  and  $C_c \in \mathcal{B}(W, Y)$ ,  $D_c \in \mathcal{B}(U, Y)$  such that  $\text{Dom}(A) \subset W \subset H$  continuously, and  $\hat{\mathcal{D}}(s) = D_c + C_c(s - A)^{-1}B$  for  $s \in \mathbb{C}_{\omega_A}^+$ .*

*Assume that  $x_0 \in H$  and  $u, x \in L_\omega^2(\mathbb{R}_+; *)$ , where  $x := \mathcal{A}x_0 + \mathcal{B}\tau u$ . Then  $y := \mathcal{C}x_0 + \mathcal{D}u \in L_\omega^2$ , and equations  $\hat{x} = (\cdot - A)^{-1}(x_0 + B\hat{u})$ ,  $\hat{y} = \hat{\mathcal{D}}_\Sigma \hat{u} + C(\cdot - A)^{-1}x_0$  hold on  $\mathbb{C}_\omega^+ \setminus \sigma(A)$  and a.e. on  $(\omega + i\mathbb{R}) \setminus \sigma(A)$ .*

(We can always take  $W := H_B := (s - A)^{-1}BU + \text{Dom}(A)$  for any  $s \in \rho(A)$  (this is independent of  $s$ ), but even so  $C_c, D_c$  need not be unique. Necessarily always  $H_B \subset W$ . See Lemma A.2(b3) for  $\omega_A$  and p. 68 for  $\hat{\mathcal{D}}_\Sigma$ .)

See, e.g., Section 6.3 of [M02] for more on  $C_c, D_c$  (e.g., Lemma 6.3.10(c) for equations (1)).

**Proof:** Combine theorems and lemmas 6.3.9, 6.3.10(a), 6.7.8, 6.3.20 and 6.2.11(c1) of [M02] to get all this with  $\omega' := \max(\omega, \omega_A)$  in the last equation. By holomorphicity (in  $H_{-1}$ ), we can extend equations  $(s - A)\hat{x}(s) = x_0 + B\hat{u}(s)$  and  $\hat{y} = C_c\hat{x} + D_c\hat{u}$  to  $\mathbb{C}_\omega^+$  and to (a.e.)  $\omega + i\mathbb{R}$  (the proof of Lemma 6.3.20 of [M02]; e.g.,  $x$  is continuous  $\mathbb{C}_\omega^+ \rightarrow W$ ). But  $D_c + C_c(s - A)^{-1}B - \hat{\mathcal{D}}(z) = (z - s)C_c(s - A)^{-1}(z - A)^{-1}B = (z - s)C(s - A)^{-1}(z - A)^{-1}B$  for  $z \in \mathbb{C}_{\omega_A}^+$ ,  $s \in \rho(A)$ , hence also the last claim holds (see Lemma A.2(c)).  $\square$

By  $\text{rconn}(V)$  we denote the “rightmost maximal connected component” of  $V$ , i.e., the maximal connected component that contains some right half-plane, provided that such exists. If  $F$  is holomorphic on  $V$ ,  $G$  is holomorphic on  $W$ , and  $F = G$  on some  $(r, +\infty)$ , then  $F = G$  on  $\text{rconn}(V \cap W)$ , by holomorphicity (when  $V, W$  are open and contain some right half-plane). This will be applied below:

**Lemma A.2** ( $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}}$ ) *Let  $\Sigma = [\frac{\mathcal{A}}{\mathcal{C}} + \frac{\mathcal{B}}{\mathcal{D}}]$  be a WPLS on  $(U, H, Y)$  and  $\omega \in \mathbb{R}$ .*

(a) *If  $\hat{\mathcal{D}}$  is holomorphic on  $\mathbb{C}_\omega^+$ , then for all  $s, z \in \text{rconn}(\rho(A) \cap \mathbb{C}_\omega^+) \supset \mathbb{C}_{\omega_A}^+$ , we have  $\hat{\mathcal{D}}_\Sigma(s) = \hat{\mathcal{D}}(s)$  and*

$$\hat{\mathcal{D}}(s) - \hat{\mathcal{D}}(z) = (z - s)C(s - A)^{-1}(z - A)^{-1}B. \quad (171)$$

(b1) *If  $\hat{\mathcal{C}}$  is holomorphic on  $\mathbb{C}_\omega^+$ , then so is  $\hat{\mathcal{D}}$ , and, for all  $s, z \in \mathbb{C}_\omega^+ \setminus \sigma(A)$  and  $s' \in \mathbb{C}_\omega^+$ , equations (171),  $\hat{\mathcal{C}}(s) = C(s - A)^{-1}$ ,  $\hat{\mathcal{D}}_\Sigma(s) = \hat{\mathcal{D}}(s)$  and  $\hat{\mathcal{D}}(s') - \hat{\mathcal{D}}(z) = (z - s')\hat{\mathcal{C}}(s')(z - A)^{-1}B$  hold.*

(b2) *If  $\hat{\mathcal{C}}$  is  $\omega$ -stable (or  $\omega'$ -stable for all  $\omega' > \omega$ ), then  $\hat{\mathcal{C}}, \hat{\mathcal{D}}$  are holomorphic on  $\mathbb{C}_\omega^+$ .*

(b3)  *$\Sigma$  is  $\alpha$ -stable for any  $\alpha > \omega_A := \inf_{t>0} [t^{-1} \log \|\mathcal{A}^t\|]$ , and  $\mathbb{C}_{\omega_A}^+ \subset \rho(A)$ .*

(c) *We have  $\hat{\mathcal{D}}_\Sigma = D_c + C_c(\cdot - A)^{-1}B$  on  $\rho(A)$ , and  $\hat{\mathcal{D}}_\Sigma \in \mathcal{H}(\rho(A); \mathcal{B}(U, Y))$ , when  $C_c, D_c$  are as in Lemma A.1.*

(d)  *$\widehat{\mathcal{A}x_0}(s) = (s - A)^{-1}x_0$  for  $s \in \mathbb{C}_{\omega_A}^+$ , and  $\widehat{\mathcal{B}\tau u}(s) = (s - A)^{-1}B\hat{u}$  for  $s \in \mathbb{C}_{\max\{\omega_A, \omega\}}^+$ , when  $x_0 \in H$ ,  $u \in L_\omega^2(\mathbb{R}_+; U)$ .*

(e) *If  $u, y \in L_\omega^2(\mathbb{R}_+; *)$ ,  $x_0 \in H$ , where  $y := \mathcal{C}x_0 + \mathcal{D}u$ , then  $\hat{y} = C(\cdot - A)^{-1}x_0 + \hat{\mathcal{D}}_\Sigma \hat{u}$  on  $\text{rconn}(\rho(A) \cap \mathbb{C}_\omega^+)$ .*

(f) *“ $\mathbb{C}_\omega^+$ ” may be replaced by “ $\mathbb{C}_\omega^+ \setminus E$ ” in (a) and (b1) if  $E$  has no limit points in  $\mathbb{C}_\omega^+$ . In particular, this applies with  $\omega = 0$  (resp. with some  $\omega < 0$ ) if  $\Sigma$  is output-stabilizable (resp. exponentially stabilizable) and  $\dim U < \infty$ .*

Here  $\rho(A) := \rho(A)$  is the resolvent set of  $A$ , and  $\rho_\infty(A) := \text{rconn}(\rho(A))$  is its maximal connected component containing  $\mathbb{C}_{\omega_A}^+$ . By  $\hat{\mathcal{C}}$  we mean the map  $\mathbb{C}_\omega^+ \rightarrow \mathcal{B}(H, Y)$  that satisfies  $\hat{\mathcal{C}}x_0 = \widehat{\mathcal{C}x_0} \forall x_0 \in H$  ( $\hat{\mathcal{C}}$  exists and is holomorphic for any  $\omega \geq \omega_A$ , by Theorem 3.10.1 of [HP57]), or its holomorphic extension to a right half-plane.

The characteristic function  $\hat{\mathcal{D}} := \hat{\mathcal{D}}_\Sigma$  of  $\Sigma$  is defined by extending the equation (171) from  $s, z \in \mathbb{C}_{\omega_A}^+$  to all  $s, z \in \rho(A)$  (cf. [SW03], Section 2). By taking adjoints, we observe that  $\hat{\mathcal{D}}_{\Sigma^d}(s) = \hat{\mathcal{D}}_\Sigma(s^*)^*$  for all  $s \in \rho(A^*) = \rho(A)^*$ .

By Example A.3, we may have  $\hat{\mathcal{D}}_\Sigma = -1$  on the unit disc even if  $\hat{\mathcal{D}} \equiv 0$  on  $\mathbb{C}$  (and hence  $\mathcal{D} = 0$ ), despite of bounded generators. In general,  $\hat{\mathcal{D}}_\Sigma$  depends on the whole realization  $\Sigma$  of  $\mathcal{D}$ , not merely on  $\mathcal{D}$  and  $A$ , hence we write  $\hat{\mathcal{D}}$  only when the realization is obvious from the context. Typically,  $\hat{\mathcal{X}} = I - \hat{\mathcal{F}}$  refers to  $[\frac{\mathcal{A}_\circ}{-\mathcal{X}} + \frac{\mathcal{B}_\circ}{\mathcal{X}}]$  (or  $\Sigma_{\text{ext}}$  for  $\hat{\mathcal{F}}$ ) and  $\hat{\mathcal{M}} = I + \hat{\mathcal{F}}_\circ$  to  $[\frac{\mathcal{A}_\circ}{\mathcal{X}_\circ} + \frac{\mathcal{B}_\circ}{-\mathcal{M}}]$  (or  $\Sigma_\circ$  for  $\hat{\mathcal{F}}_\circ$ ).

**Proof of Lemma A.2:** (a) See, e.g., [M02], Lemma 6.2.11(d2) for  $s, z \in \mathbb{C}_{\omega_A}^+$ . Since both sides are holomorphic on the connected set  $\text{rconn}(\rho_\infty(A) \cap \mathbb{C}_\omega^+)$ , (a) holds.

(b3)&(d) See, e.g., Lemma 6.1.10(a2) and Theorem 6.2.11 of [M02].

(b2) By Lemma F.3.2(d) of [M02], there is  $\hat{\mathcal{C}} \in H_{\text{strong}}^2(\mathbb{C}^+; \mathcal{B}(H, Y))$  s.t.  $\widehat{\mathcal{C}}x = \hat{\mathcal{C}}x$  on  $\mathbb{C}_\omega^+$  for all  $x \in H$ . By Lemmas 6.1.11 and 6.2.1,  $\hat{\mathcal{D}} \in H(\mathbb{C}_\omega^+; \mathcal{B}(H, Y))$ . (Similarly,  $\mathcal{C} \in \mathcal{B}(H, L_{\omega'}^2)$  implies that  $\hat{\mathcal{C}}, \hat{\mathcal{D}} \in H(\mathbb{C}_{\omega'}^+, \mathcal{B})$ ; if this holds for all  $\omega' > \omega$ , then  $\hat{\mathcal{C}}, \hat{\mathcal{D}} \in H(\mathbb{C}_\omega^+, \mathcal{B})$ .)

(c)  $D_c + C_c(s - A)^{-1}B - \hat{\mathcal{D}}(z) = (z - s)C_c(s - A)^{-1}(z - A)^{-1}B = (z - s)C(s - A)^{-1}(z - A)^{-1}B$  for  $z \in \mathbb{C}_{\omega_A}^+$ ,  $s \in \rho(A)$ . (Here  $(C_c, D_c)$  may be any compatible pair for  $\Sigma$ .) From the definition (see (171)) we observe that  $\hat{\mathcal{D}}$  is holomorphic (use Lemma A.4.4(a) of [M02]).

(b1) 1° “ $\mathcal{C}$ ”: We have  $\hat{\mathcal{C}}(s)(s - A) = C$  on  $\mathbb{C}_{\omega_A}^+ \cap \mathbb{C}_\omega^+$ , hence on  $\mathbb{C}_\omega^+$ , by holomorphicity (see Lemma A.4.4(b) of [M02]), hence  $\hat{\mathcal{C}}(s) = C(s - A)^{-1}$  on  $\mathbb{C}_\omega^+ \setminus \sigma(A)$ .

2° “ $s'$  and (171)”: Set  $\alpha := \max\{\omega, \omega_A\}$ . By (a) and 1°, we have  $\hat{\mathcal{D}}(s) - \hat{\mathcal{D}}(z) = (z - s)\hat{\mathcal{C}}(s)(z - A)^{-1}B$  for all  $s, z \in \mathbb{C}_\alpha^+$ , hence for all  $s \in \mathbb{C}_\omega^+$ ,  $z \in \mathbb{C}_\alpha^+$  (since this equation specifies a (unique) holomorphic extension of  $\hat{\mathcal{D}}(s)$  to  $s \in \mathbb{C}_\omega^+$ ; recall that we identify a function with extensions to any right half-planes). By 1°, this leads to (171) for all  $s \in \mathbb{C}_\omega^+ \setminus \sigma(A)$ ,  $z \in \mathbb{C}_\alpha^+$ . Substitute it to  $[\hat{\mathcal{D}}(s') - \hat{\mathcal{D}}(z)] - [\hat{\mathcal{D}}(s) - \hat{\mathcal{D}}(z)]$  to obtain (171) for  $s, z \in \mathbb{C}_\omega^+ \setminus \sigma(A)$  (use the resolvent equation  $(z - s)(s - A)^{-1}(z - A)^{-1} = (s - A)^{-1} - (z - A)^{-1}$ ).

3° “ $\mathcal{D}(s) = \hat{\mathcal{D}}_\Sigma(s)$ ”: Fix  $z \in \mathbb{C}_{\omega_A}^+$ . By the definition of  $\hat{\mathcal{D}}_\Sigma$ , we have  $\hat{\mathcal{D}} - \hat{\mathcal{D}}_\Sigma = 0$  on  $\mathbb{C}_{\omega_A}^+$ ; by definition and 2°, we have  $\hat{\mathcal{D}}_\Sigma(s) - \hat{\mathcal{D}}(z) = (z - s)C(s - A)^{-1}(z - A)^{-1} = \hat{\mathcal{D}}(s) - \hat{\mathcal{D}}(z)$  for  $s \in \mathbb{C}_\omega^+ \setminus \sigma(A)$ .

(e) Now  $\hat{y} \in H^2(\mathbb{C}_\omega^+; Y)$  and  $C(\cdot - A)^{-1}x_0 + \hat{\mathcal{D}}_\Sigma \hat{u} \in H(\rho(A); Y)$ , and  $\hat{y} = \hat{\mathcal{C}}x_0 + \hat{\mathcal{D}}\hat{u} = C(\cdot - A)^{-1}x_0 + \hat{\mathcal{D}}_\Sigma \hat{u}$  on  $\mathbb{C}_\omega^+ \cap \mathbb{C}_{\omega_A}^+ \subset \text{rconn}(\rho(A) \cap \mathbb{C}_\omega^+)$ .

(f) The same proofs still apply for (a) and (b1). As noted below Corollary 5.10,  $\hat{\mathcal{C}}$  and  $\hat{\mathcal{D}}$  are meromorphic on  $\mathbb{C}^+ \setminus E$  (resp.  $\mathbb{C}_\omega^+ \setminus E$  for some  $\omega < 0$ ) when  $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset$  (resp.  $\mathcal{U}_{\text{exp}}(x_0) \neq \emptyset$ ) for all  $x_0 \in H$ .  $\square$

**Example A.3** ( $\mathcal{D} = 0$  but  $\hat{\mathcal{D}}_\Sigma = -1$  on the unit disc). In the example on p. 843 of [W94a] we have  $H = \ell^2(\mathbb{Z})$ ,  $U = Y = \mathbb{C}$ ,  $A$  is the right shift  $Ae_k = e_{k+1}$ ,  $B = e_1$ ,  $C = \langle \cdot, e_0 \rangle$ ,  $D = 0$ , where  $\{e_k\}_{k \in \mathbb{Z}}$  is the natural base of  $H$ .

Thus,  $A, B, C, D$  are bounded,  $\sigma(A) = \sigma(A^{-1})$  is the unit circle,  $\hat{\mathcal{D}} \equiv 0$  (hence  $\mathcal{D} = 0$ ) but  $\hat{\mathcal{D}}_\Sigma = -1$  on the unit disc.  $\triangleleft$

A transfer function ( $\hat{\mathcal{D}}$ ) is uniquely defined by the I/O map ( $\mathcal{D}$ ) and vice versa. We can allow for a holomorphic extension of  $\hat{\mathcal{D}}$  and still keep uniqueness if we require the domain to be a half-plane with a discrete set of singularities:

**Remark A.4 (Transfer function ( $\hat{\mathcal{D}}$ ))** We define a transfer function (see Theorem 2.5) on the “maximal half-plane of discrete meromorphicity”, i.e., on  $\mathbb{C}_\omega^+ \setminus E$ , if  $\omega \in \mathbb{R}$ ,  $E$  is discrete on  $\mathbb{C}_\omega^+$  (i.e., has no limit points on  $\mathbb{C}_\omega^+$ ) and  $\hat{\mathcal{D}}$  has a holomorphic extension onto  $\mathbb{C}_\omega^+ \setminus E$  (as in (f) above; similarly for  $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{T}}, \hat{\mathcal{C}}$ ).

Obviously, this defines  $\hat{\mathcal{D}}$  uniquely. It has been thought that  $\mathbb{C}_\omega^+ \setminus E$  being connected would suffice, but that is not true, as shown in Example A.5, where  $E = \sigma(A)$  is a half-line, so that  $\rho(A)$  is connected but yet the value of the characteristic function  $\hat{\mathcal{D}}_\Sigma$  on  $\mathbb{C}^-$  depends on the realization of  $\hat{\mathcal{D}} = 1/\sqrt{s}$ .

As explained above, by taking two different branches of (the transfer function  $\hat{\mathcal{D}}(s) := 1/\sqrt{s}$ ), we can have different values of  $\hat{\mathcal{D}}_\Sigma(-1)$ , even for characteristic functions of real-

izations of  $\hat{\mathcal{D}}$  (hence holomorphic extensions of  $\hat{\mathcal{D}}|_{\mathbb{C}^+}$ ) whose resolvent sets are connected:

**Example A.5** (Transfer function cannot be uniquely extended around nondiscrete sets, not even for connected  $\rho(A)$ ). Let  $\Sigma_1, \Sigma_2$  be realizations of  $s \mapsto 1/\sqrt{s}$  (the primary branch on  $\mathbb{C}^+$ ) with  $\sigma(A_1) = i\mathbb{R}_+$ ,  $\sigma(A_2) = -i\mathbb{R}_+$ . Then  $\hat{\mathcal{D}}_1 = \hat{\mathcal{D}}_2$  on  $\mathbb{C}^+$  but  $\hat{\mathcal{D}}_{\Sigma_1}(-1) = -i$ ,  $\hat{\mathcal{D}}_{\Sigma_2}(-1) = i$ . Moreover,  $-1 \in \rho_\infty(A_k) = \rho(A_k)$  ( $k = 1, 2$ ).  $\triangleleft$

**Proof:** A realization of  $\hat{\mathcal{D}}(s) = 1/\sqrt{s}$  with  $\sigma(A) = \mathbb{R}_-$  is given in [O96]. Since  $\hat{\mathcal{D}}(+\infty) = 0$  exists,  $\hat{\mathcal{D}}$  is regular. Set  $\tilde{A} := -iA$ ,  $\tilde{B} := -\sqrt{i}B$  to obtain  $\hat{\tilde{\mathcal{D}}}(s) = C_w(s + iA)^{-1}(-\sqrt{i})B = \sqrt{i}C_w(is - A)^{-1}B = \sqrt{i}\hat{\mathcal{D}}(is)$ , which is obviously holomorphic outside  $i\mathbb{R}_+$  and is a branch of  $1/\sqrt{s}$  (since they obviously coincide on  $\mathbb{R}_+$ ). Thus, we have obtained the realization  $\Sigma_1 := \begin{bmatrix} \tilde{A} & \tilde{B} \\ C & 0 \end{bmatrix}$  of  $\tilde{\mathcal{D}} := \sqrt{i}\hat{\mathcal{D}}$ . Similarly, one obtains  $\Sigma_2$  and the rest is straight-forward.  $\square$

The standard formulas of (27) also apply to controlled WPLS forms:

**Theorem A.6** ( $\widehat{\Sigma}_0$ ) Let  $\mathcal{K}_0$  be a control for  $\Sigma$  in WPLS form, and choose  $(C_c, D_c)$  as in Lemma A.1. Then  $A_0 = A + BK_0$  and  $C_0 = C_c + D_cK_0$  on  $\text{Dom}(A_0)$ ,  $(s - A_0)^{-1} - (s - A)^{-1} = (s - A)^{-1}BK_0(s - A_0)^{-1}$ , and  $C_0(s - A_0)^{-1} = C(s - A)^{-1} + \hat{\mathcal{D}}_\Sigma(s)K_0(s - A_0)^{-1}$  for all  $s \in \rho(A) \cap \rho(A_0)$ .

If  $\mathcal{K}_0$  is  $\omega$ -stable, then  $\sigma(A_0) \cap \overline{\mathbb{C}_\omega^+} \subset \sigma(A)$  and  $\sigma_p(A_0) \cap \overline{\mathbb{C}_\omega^+} \subset \sigma_p(A)$ .

In particular, output-stabilizing feedback does not add unstable spectrum ( $\sigma(A_0) \cap \overline{\mathbb{C}_\omega^+}$ ). Actually, we prove the stronger claim  $\rho(A) \cap \widehat{\rho(\mathcal{K}_0)} \subset \rho(A_0)$  in 2° below. Moreover, only the eigenvalues ( $\sigma_p(A) := \{s \in \mathbb{C} \mid (s - A)x_0 = 0 \text{ for some } x_0 \in H \setminus \{0\}\}$ ) of  $A$  may be those of  $A_0$  on  $\overline{\mathbb{C}_\omega^+}$ . See also Lemma 3.3.

By duality (see Lemma 3.4),  $\sigma(A^*) \cap \overline{\mathbb{C}_\omega^+} \subset \sigma(A_0^*)$  and  $\sigma_p(A^*) \cap \overline{\mathbb{C}_\omega^+} \subset \sigma_p(A_0^*)$  if  $\mathcal{B}$  is  $\omega$ -stable. Recall that  $\sigma(A^*) = \sigma(A)^*$ .

**Proof:** 1° See Lemma 8.3.17(a) of [M02] for  $A_0$  and  $C_0$ ; here  $(C_c, D_c)$  is any compatible pair for  $\Sigma$ . From (8.61) and (8.63)–(8.64) of [M02] (in (8.63) “ $C_c$ ” should be “ $+C_c$ ”), we get the other two equations.

2° Assume that  $\Omega \subset \mathbb{C}$  is open and connected and contains some right half-plane. Assume that  $\widehat{\mathcal{K}}_0$  has a holomorphic extension  $\Omega \rightarrow \mathcal{B}(H, U)$ . Then  $\Omega' := \rho(A) \cap \Omega \subset \rho(A_0)$  and  $(s - A_0)^{-1} = (s - A)^{-1}[I + B\widehat{\mathcal{K}}_0(s)] =: f(s)$  on  $\Omega'$ : Fix  $z > \max\{\omega_A, \omega_{A_0}\}$ . Since  $(z - A_0)^{-1} = f(z) \in \mathcal{B}(H)$  is one-to-one, our claim follows from Lemma B.6 once we have (178). Set  $R_s := (s - A)^{-1}$  to have, for any  $s \in \Omega'$ , that

$$f(s) - f(z) = (R_s - R_z)[I + B\widehat{\mathcal{K}}_0(z)] + R_s B[\widehat{\mathcal{K}}_0(s) - \widehat{\mathcal{K}}_0(z)],$$

and  $f(s)f(z) = R_s R_z[I + B\widehat{\mathcal{K}}_0(z)] + R_s B\widehat{\mathcal{K}}_0(s)R_z[I + B\widehat{\mathcal{K}}_0(z)]$ . By these and the Resolvent equation,  $f(s) - f(z) - (z - s)f(s)f(z) = R_s BW$ , where

$$W := \widehat{\mathcal{K}}_0(s) - \widehat{\mathcal{K}}_0(z) - (z - s)\widehat{\mathcal{K}}_0(s)R_z[I + B\widehat{\mathcal{K}}_0(z)],$$

hence  $W(z - A_0) = \widehat{\mathcal{K}}_0(s)(z - A_0) - K_0 - (z - s)\widehat{\mathcal{K}}_0(s) = \widehat{\mathcal{K}}_0(s)(s - A_0) - K_0 = 0$  on  $\text{Dom}(A_0)$ . (We used here the fact that  $\widehat{\mathcal{K}}_0(s)(s - A_0) = K_0$  for  $s > z$ , hence for any  $s \in \Omega'$ .) Since  $\text{Ran}(z - A_0) = H$ , equation (178) holds and we are done.

3° Case  $\mathcal{K}_0$   $\omega$ -stable,  $\sigma(A_0)$ : Let  $z \in \rho(A) \cap \overline{\mathbb{C}_\omega^+}$ ; we should show that  $z \in \rho(A_0)$ . W.l.o.g.,  $\omega \leq 0$  and  $z = 0$  (replace  $A$  by  $A - z$  as in Lemma 6.1.9 of [M02]). We have  $\mathbb{C}_\omega^+ \cap \rho(A) \subset \rho(A_0)$ , by 2°, hence  $s \in \rho(A_0)$  for small  $s > 0$ . Since  $\widehat{\mathcal{K}}_0 \in H_{\text{strong}}^2$ , we have  $s\widehat{\mathcal{K}}_0(s) \rightarrow 0$  (uniformly), as  $s \rightarrow 0+$ , by Lemma B.8. But  $(s - A)^{-1} \rightarrow A^{-1}$  and  $(s - A)^{-1}B \rightarrow A^{-1}B$ , hence  $s(s - A_0)^{-1} \rightarrow 0$ , hence  $0 \notin \sigma(A_0)$ , by Lemma B.7.

4° Case  $\mathcal{K}_0$   $\omega$ -stable,  $\sigma_p(A)$ : As in 3°, assume now that  $0 \in \overline{\mathbb{C}_\omega^+} \setminus \sigma_p(A)$ ,  $\omega \leq 0$ . If  $x_0 \in \text{Dom}(A_0)$  and  $A_0 x_0 = 0$ , then  $\mathcal{A}_0 x_0 \equiv x_0$ , hence  $\mathcal{K}_0 x_0 \equiv K_0 x_0$ , hence  $K_0 x_0 = 0$  (since  $\mathcal{K}_0 \in L^2$ ), hence  $Ax_0 = A_0 x_0 + BK_0 x_0 = 0$ , hence  $(x_0 \in \text{Dom}(A) \text{ and } x_0 = 0$ .  $\square$

Next we give similar but stronger results for (well-posed) state feedback, i.e., equation (25) in the frequency domain. By Lemma A.2 (and Definition 3.5), both sides of (172) equal the Laplace transforms of the components of  $\Sigma_{\circ}$  on some right half-plane (see Proposition 6.6.18 of [M02] for details and further results). By Example A.3, the Laplace transforms need not equal (172) outside  $\rho_{\infty}(A)$ . Nevertheless, (172) itself holds wherever both sides are defined:

**Theorem A.7** ( $\rho(A) \setminus \sigma(\hat{\mathcal{M}}) \subset \rho(A_{\circ})$  &  $\hat{\mathcal{M}}_{\Sigma_{\circ}} = \hat{\mathcal{X}}_{\Sigma}^{-1}$ ) Let  $[\mathcal{K} \mid \mathcal{F}]$ ,  $\mathcal{M}$  and  $\Sigma = [\frac{\mathcal{A}}{\mathcal{C}} \mid \frac{\mathcal{B}}{\mathcal{D}}]$  be as in Definition 3.5, and set  $\mathcal{X} := I - \mathcal{F}$  ( $= \mathcal{M}^{-1}$ ). Let  $s \in \rho(A)$ . Then  $s \in \rho(A_{\circ}) \Leftrightarrow \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s) \in \mathcal{GB}(U)$  ( $\Leftarrow \operatorname{Re} s \geq \omega$  if  $\mathcal{K}_{\circ}$  or  $\mathcal{B}_{\circ}$  is  $\omega$ -stable). Moreover, for all  $s \in \rho(A) \cap \rho(A_{\circ})$ , we have

$$\begin{aligned} & \left[ \begin{array}{c|c} (s - A_{\circ})^{-1} & (s - A_{\circ})^{-1}B_{\circ} \\ \hline C_{\circ}(s - A_{\circ})^{-1} & \widehat{(\mathcal{D}_{\circ})}_{\Sigma_{\circ}}(s) \\ K_{\circ}(s - A_{\circ})^{-1} & \hat{\mathcal{M}}_{\Sigma_{\circ}}(s) \end{array} \right] \\ &= \left[ \begin{array}{c|c} (s - A)^{-1}[I + BK_{\circ}(s - A_{\circ})^{-1}] & (s - A)^{-1}B\hat{\mathcal{M}}_{\Sigma_{\circ}}(s) \\ \hline C(s - A)^{-1} + \hat{\mathcal{D}}_{\Sigma}(s)\hat{\mathcal{M}}_{\Sigma_{\circ}}(s)K(s - A)^{-1} & \hat{\mathcal{D}}_{\Sigma}(s)\hat{\mathcal{M}}_{\Sigma_{\circ}}(s) \\ \hat{\mathcal{M}}_{\Sigma_{\circ}}(s)K(s - A)^{-1} & \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s)^{-1} \end{array} \right]. \end{aligned} \quad (172)$$

Observe that above we refer to the characteristic functions of these specific realizations (and  $\hat{\mathcal{M}}_{\Sigma_{\circ}} := I + (\mathcal{F}_{\circ})_{\Sigma_{\circ}}$ ). E.g., if  $[\frac{\mathcal{A}}{\mathcal{C}} \mid \frac{\mathcal{B}}{\mathcal{D}}]$  the system of Example A.3 but with  $F := 1/2$ , then  $\hat{\mathcal{F}} \equiv 1/2 \equiv \hat{\mathcal{X}}$  on  $\mathbb{C}$ , hence  $\hat{\mathcal{M}} \equiv 2$ , but  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}} = 3/2 = \hat{\mathcal{M}}_{\Sigma_{\circ}}^{-1}$  on the unit disc, whereas  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}} \equiv 1/2 \equiv \hat{\mathcal{M}}_{\Sigma_{\circ}}^{-1}$  for the alternative realization  $[\frac{\mathcal{A}}{\mathcal{C}} \mid \frac{\mathcal{B}}{\mathcal{D}}] := \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}$ .

Exchange the roles of  $\Sigma$  and  $[\frac{\mathcal{A}_{\circ}}{\mathcal{C}_{\circ}} \mid \frac{\mathcal{B}_{\circ}}{\mathcal{D}_{\circ}}]$ , to obtain further formulas and implications (with the roles of  $\pm [\mathcal{K} \mid \mathcal{F}]$  and  $\mp [\mathcal{K}_{\circ} \mid \mathcal{F}_{\circ}]$  exchanged, as in Lemma 6.6.14 of [M02]). Also Theorem A.6 applies both ways.

**Proof of Theorem A.7:** Set  $R_s := (s - A)^{-1}$  ( $s \in \rho(A)$ ),  $T_s := (s - A_{\circ})^{-1}$  ( $s \in \rho(A_{\circ})$ ).

I Let  $s \in \rho(A)$ . Let  $(K_c, F_c)$  be a compatible pair for  $[\frac{\mathcal{A}}{\mathcal{C}} \mid \frac{\mathcal{B}}{\mathcal{D}}]$ ,  $X_c := I - F_c$ . Set  $X := \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s) = X_c - K_c(s - A)^{-1}B$ .

1° Assume that  $M := X^{-1} \in \mathcal{GB}(U)$  exists. We shall show that  $T = (s - A_{\circ})^{-1}$ , where  $T := (I + R_s B M K)R_s \in \mathcal{B}(H)$  (actually,  $T \in \mathcal{B}(H, H_B)$ ).

By Theorem A.6 and, e.g., the proof of (b1) of Proposition 6.6.18 of [M02], we have  $A_{\circ} = A + BK_{\circ}$  and  $X_c K_{\circ} = K_c$  on  $\operatorname{Dom}(A_{\circ})$ . Consequently, on  $\operatorname{Dom}(A_{\circ})$  we have  $T(s - A_{\circ}) = (I + R_s B M K_c)R_s(s - A - BK_{\circ}) = (I + R_s B M K_c)(I - R_s B K_{\circ}) = I + R_s B M [K_c - X K_{\circ} - K_c R_s B K_{\circ}] = I + R_s B M [K_c - X_c K_{\circ}] = I$ .

By the dual result, also  $(\bar{s} - A_{\circ}^*) = (s - A_{\circ})^*$  has a bounded left-inverse, hence  $s - A_{\circ}$  has a bounded right-inverse, hence  $s \in \rho(A_{\circ})$  (because  $s - A_{\circ}$  is also closed, densely defined and one-to-one).

2° Assume that  $s \in \rho(A) \cap \rho(A_{\circ})$ . Then  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s) = \hat{\mathcal{M}}_{\Sigma_{\circ}}(s)^{-1}$ , by part II of this proof, hence then  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s) \in \mathcal{GB}(U)$ .

3° “ $\Leftarrow \operatorname{Re} s \geq \omega$ ”: This follows from Theorem A.6 (use duality for  $\mathcal{B}_{\circ}$ ).

II Choose some  $z > \max\{\omega_A, \omega_{A_{\circ}}\}$ . Fix  $s \in \rho(A) \cap \rho(A_{\circ})$ .

1° “ $\hat{\mathcal{M}}_{\Sigma_{\circ}}$ ”:  $\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s) := \hat{\mathcal{X}}(z) + (z - s)K R_s R_z B$  (see p. 68), and  $\hat{\mathcal{M}}_{\Sigma_{\circ}}(s) := \hat{\mathcal{M}}(z) + (z - s)K_{\circ} T_s T_z B_{\circ}$ .

Therefore,  $S := \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s)\hat{\mathcal{M}}_{\Sigma_{\circ}}(s) = I + (z - s)\hat{\mathcal{X}}(z)K_{\circ} T_s T_z B_{\circ} + (z - s)K R_s R_z B \hat{\mathcal{M}}(z) + (z - s)^2 K R_s R_z B K_{\circ} T_s T_z B_{\circ}$ . But  $K_{\circ}(z - A_{\circ})^{-1} = \mathcal{K}_{\circ}(z)$ , and  $\hat{\mathcal{X}}(z)\widehat{\mathcal{K}}_{\circ}(z) = \mathcal{K}(z)$  (by (6.133) of [M02]); similarly,  $R_z B \hat{\mathcal{M}}(z) = \widehat{\mathcal{B}}_{\circ}(z) = (z - A_{\circ})^{-1}B_{\circ}$ , and, by (6.137),  $R_z B K_{\circ} T_z = T_z - R_z$  hence  $S - I = (z - s)K V B_{\circ}$ , where  $V = -R_z T_s + R_s T_z + (z - s)R_s(T_z - R_z)T_s$ . By the resolvent equation,  $V = 0$ , hence  $S = I$ . Similarly,  $\hat{\mathcal{M}}_{\Sigma_{\circ}}(s)\hat{\mathcal{X}}_{\Sigma_{\text{ext}}}(s) = \dots = I$ .

2° “ $\widehat{\mathcal{B}}_{\circ}, \widehat{\mathcal{K}}_{\circ}, \widehat{\mathcal{A}}_{\circ}$ ”: Apply Theorem A.6 to  $[\mathcal{K} \mid -\mathcal{X}]$  in place of  $[\mathcal{C} \mid \mathcal{D}]$  to obtain that  $0(s - A_0)^{-1} = K R_s - \hat{\mathcal{X}}_{\Sigma}(s)K_0 T_s$ , i.e.,  $K R_s = \hat{\mathcal{X}}_{\Sigma}(s)K_0 T_s$ . By 1°, this

equals  $\hat{\mathcal{M}}_{\Sigma_{\odot}}(s)KR_s = K_0T_s$ , as claimed. The formula for  $(s - A_{\odot})^{-1}B_{\odot}$  follows by duality.

3° “ $\widehat{\mathcal{C}}_{\odot}, \widehat{\mathcal{D}}_{\odot}$ ”: Apply the above for  $[\mathcal{X} | \mathcal{Z}]$  in place of  $[\mathcal{K} | \mathcal{F}]$  to obtain that  $\begin{bmatrix} I & \hat{\mathcal{D}}_{\Sigma} \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}^{-1} \\ 0 & \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}^{-1} \end{bmatrix} = \begin{bmatrix} I & (\hat{\mathcal{D}}_{\odot})_{\Sigma_{\odot}} \hat{\mathcal{M}}_{\Sigma_{\odot}} \\ 0 & \hat{\mathcal{M}}_{\Sigma_{\odot}} \end{bmatrix}$  and  $\begin{bmatrix} C_{\odot} \\ K_{\odot} \end{bmatrix} (s - A_{\odot})^{-1} = \begin{bmatrix} C + (\hat{\mathcal{D}}_{\odot})_{\Sigma_{\odot}} \hat{\mathcal{M}}_{\Sigma_{\odot}} K \\ \hat{\mathcal{M}}_{\Sigma_{\odot}} K \end{bmatrix} (s - A)^{-1}$ .  $\square$

If  $\sigma(A)$  is nice and  $\dim U < \infty$ , then any I/O-stabilizing state-feedback pair allows for a holomorphic extension of  $\hat{\mathcal{N}}, \hat{\mathcal{M}}$  over the imaginary axis:

**Lemma A.8** *Use notation of Definition 3.5,  $[\mathcal{K} | \mathcal{F}]$  being admissible. If  $\mathcal{N}, \mathcal{M}$  are stable,  $\dim U < \infty$ ,  $\sigma(A) \cap i\mathbb{R}$  consists of isolated poles, and  $\inf \rho_{\infty}(A) \leq 0$ , then there is an open  $\Omega \subset \mathbb{C}$  s.t.  $\mathbb{C}^+ \cup (\rho_{\infty}(A) \setminus Z_g) \subset \Omega$  and  $\hat{\mathcal{N}}, \hat{\mathcal{M}}$  have holomorphic extensions to  $\Omega$ . Moreover, then  $\rho_{\infty}(A) \setminus Z_g \subset \rho_{\infty}(A_{\odot})$ , and the only nonremovable singularities of (172) on  $\rho_{\infty}(A)$  are isolated poles.*

Here  $Z_g := \{s \in \rho_{\infty}(A) \mid \det \hat{\mathcal{X}}_{\Sigma_{\text{ext}}} = 0\}$  (it is discrete). The requirement  $\inf \rho_{\infty}(A) \leq 0$  means that you can reach the imaginary axis from  $+\infty$  through  $\rho(A)$ . By the proof (6°), the extensions equal the characteristic functions on  $\rho_{\infty}(A) \setminus Z_g$ , which contains almost every point of  $i\mathbb{R}$ .

For this kind of systems, “q.r.c.” and “r.c.” are equivalent in the sense of Lemma 5.12.

**Proof:** 1° For each  $t \in \mathbb{R}$ , choose a maximal disc (punctured if necessary) with center  $it$  and contained in  $\rho(A)$ . Let  $G$  be the union of these discs. Obviously,  $G$  and  $G \cap \mathbb{C}^+$  are connected open subsets of  $\rho(A)$ , and  $i\mathbb{R} \setminus \sigma(A) \subset G$ .

2° We have  $G \subset \rho_{\infty}(A)$ , hence  $i\mathbb{R} \setminus \sigma(A) \subset \rho_{\infty}(A)$  (because  $\inf \rho_{\infty}(A) \leq 0$  implies that  $G \cap \rho_{\infty}(A) \neq \emptyset$ ).

3°  $\Omega_1 := \rho_{\infty}(A) \cap \mathbb{C}^+$  is connected: If  $\Omega_1 \subset V \cup W$ ,  $V \cap W = \emptyset$ , and, w.l.o.g.,  $G \cap \mathbb{C}^+ \subset V$ , then (the boundary)  $\partial W \subset \mathbb{C}^+$ , hence  $\partial W \cap \rho_{\infty}(A) = \emptyset$ , hence  $W$  is a component of  $\rho_{\infty}(A)$  or  $W = \emptyset$ , QED.

4°  $g^{-1} = \hat{\mathcal{M}}$  on  $\Omega_1$ : By Theorem 2.5,  $\hat{\mathcal{N}}, \hat{\mathcal{M}} \in H^{\infty}(\mathbb{C}^+; \mathcal{B})$ . Let  $f := \hat{\mathcal{D}}_{\Sigma}$ ,  $g := \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}$ , so that  $f, g \in H(\rho(A); \mathcal{B})$ . Since  $\det g \neq 0$  on  $\rho_{\infty}(A)$ , the set  $Z_g := \{s \in \rho_{\infty}(A) \mid \det g(s) = 0\}$  is discrete in  $\rho_{\infty}(A)$ . By continuity,  $g^{-1} = \hat{\mathcal{M}}$  and  $fg^{-1} = \hat{\mathcal{N}}$  on the whole  $\Omega_1$ .

5°  $\hat{\mathcal{M}}, \hat{\mathcal{N}}$  have unique holomorphic extensions to  $\Omega$ : Because  $\Omega_2 := \rho_{\infty}(A) \cup \mathbb{C}^+ \setminus Z_g$  is open and connected, and  $fg^{-1}, g^{-1}$  are holomorphic on  $\rho_{\infty}(A) \setminus Z_g$ ,  $\hat{\mathcal{N}}, \hat{\mathcal{M}}$  have unique holomorphic extensions  $\hat{\mathcal{N}}_e, \hat{\mathcal{M}}_e : \Omega_2 \rightarrow \mathcal{B}$ . Also  $\Omega := \Omega_2 \cup i\mathbb{R}$  is open and connected. By the assumption,  $f, g$  do not have essential singularities on  $i\mathbb{R}$ , hence so do not  $\hat{\mathcal{N}}_e, \hat{\mathcal{M}}_e$ ; but they cannot have poles either, because they are bounded on  $\mathbb{C}^+$ , hence their singularities on  $i\mathbb{R}$  are removable, QED.

6° We have  $\rho_{\infty}(A) \setminus Z_g \subset \rho(A_{\odot})$ : By Theorem A.7,  $V := \rho_{\infty}(A) \setminus Z_g \subset \rho(A_{\odot})$  (hence  $V \subset \rho_{\infty}(A_{\odot})$ , hence  $G \subset \rho_{\infty}(A_{\odot})$ ), and  $\hat{\mathcal{N}}_e, \hat{\mathcal{M}}_e$  coincide with the characteristic functions of  $\hat{\mathcal{N}}, \hat{\mathcal{M}}$  on  $V$ .

7° *Nonremovable singularities*: The functions  $(s - A)^{-1}, C(s - A)^{-1}, K(s - A)^{-1}, (s - A)^{-1}B, \hat{\mathcal{D}}_{\Sigma}, \hat{\mathcal{X}}_{\Sigma_{\text{ext}}}$  are holomorphic on  $\rho(A)$  and have at most isolated poles at those of  $(s - A)^{-1}$  (by the resolvent equation: e.g.,  $(s - A)^{-1}B = (s_0 - s)(s - A)^{-1}(s_0 - A)^{-1}B$  and  $(s_0 - A)^{-1}B \in \mathcal{B}(U, H)$ ). But  $g^{-1}$  has at most isolated poles on  $\rho_{\infty}(A)$ , hence so do also the other elements of both sides of (172), since they are obtained from the functions mentioned above through sums and products.  $\square$

**Notes for Appendix A:** In 1997, the author defined compatible pairs (under a different name) and developed state-feedback and Riccati equation theories for them. The concept was further developed by O. Staffans and G. Weiss, including the existence for all WPLSs (see p. 202 of [M02]); the second paragraph of Lemma A.1 seems to be new (although partially in [M02]). Also part of Lemma A.2 is known, due to Weiss, Staffans and others; see, e.g., [S04] or Chapter 6 of [M02] for further notes and details.

Example A.3 is due to Hans Zwart (p. 843 of [W94a]). Example A.5 was constructed in correspondence with Olof Staffans. The rest of this appendix seems to be new.

The importance of characteristic functions and exact domains of transfer functions requires some explanation: Recently, the so called reciprocal system theory has been



used as a powerful tool to simplify WPLSs theory and AREs, due to Ruth Curtain and others (see, e.g., [C03]). Late 2002, we pointed out that one must use the characteristic function instead of the transfer function in the reciprocal systems theory (and suggested a separate name and symbol, i.e.,  $\hat{\mathcal{D}}_\Sigma$  instead of  $\hat{\mathcal{D}}$ ). Moreover, the reciprocal equations are justified on  $\text{rconn}(\rho(A) \cap \mathbb{C}^+)$  only, not on  $\rho(A)$  or  $\rho_\infty(A)$  in general, which restricts the applicability of the theory, although one can often circumvent these problems, as shown in [M03b] (and [M03]). These findings, thereafter widely spread, were our motivations behind Lemma A.2, Example A.5 and Theorem A.7. Later we observed these results useful also in the standard RE theory.

## B Laplace transforms

In this section we list some results on the Laplace transform (see (17)). Here  $X$  stands for an arbitrary Banach space.

**Lemma B.1** ( $s\hat{f}(s) \rightarrow f(0)$ ) *If  $f \in L_\omega^1(\mathbb{R}_+; X)$  is continuous at 0, and  $\omega \in \mathbb{R}$ , then  $s\hat{f}(s) \rightarrow f(0)$  as  $s \rightarrow +\infty$ .*

**Proof:**  $\|s\hat{f}(s) - sf(0)/s\| = \|s \int_0^\infty [f(t) - f(0)]e^{-st} dt\| = \|s \int_0^\delta + s \int_\delta^\infty\| \leq \epsilon/2 + |se^{-(s-\omega)\delta}| \|f\|_{L_\omega^1} + |e^{-s\delta}| \|f(0)\| < \epsilon$  for  $\delta > 0$  small and  $s$  big enough.  $\square$

**Lemma B.2** *If  $f \in L_\omega^2(\mathbb{R}_+; X)$ , then  $\hat{f}'(s) = s\hat{f}(s) - f(0) \forall s \in \mathbb{C}_\omega^+$ .*

(This follows by, e.g., partial integration for  $f \in W_\omega^{1,2}$ . This is the obvious extension (sometimes even the definition) of the (Sobolev) distribution derivative.)

Knowing  $\langle \hat{f}(s), \hat{g}(s) \rangle$  on a half-plane  $\mathbb{C}_\omega^+$  characterizes  $\langle f(t), g(t) \rangle$  on  $\mathbb{R}_+$ :

**Lemma B.3** ( $\langle \hat{f}, \hat{g} \rangle_H = 0 \Rightarrow \langle f, g \rangle_H = 0$  a.e.) *Let  $f, g \in L_\omega^2(\mathbb{R}_+; H)$ ,  $F, G \in L_\omega^2(\mathbb{R}_+; Y)$ ,  $\omega \leq \alpha < \beta$ . If  $\langle \hat{f}(s), \hat{g}(s) \rangle_H = \langle \hat{F}(s), \hat{G}(s) \rangle_Y$  for  $s \in \mathbb{C}_\alpha^+ \setminus \mathbb{C}_\beta^+$ , then  $\langle f(t), g(t) \rangle_H = \langle F(t), G(t) \rangle_Y$  for a.e.  $t \geq 0$ .*

**Proof:** Take  $F = 0 = G$  and  $\omega = \alpha = 0$  w.l.o.g. (use  $f \mapsto e^{-\alpha \cdot} \begin{bmatrix} f \\ F \end{bmatrix}$ ,  $g \mapsto e^{-\alpha \cdot} \begin{bmatrix} g \\ -G \end{bmatrix}$ ,  $\omega \mapsto \alpha$ ). Set  $h(t) := \langle f(t), g(t) \rangle_H$ , so that  $h \in L^1(\mathbb{R}_+; H)$ . Then  $h(t) = 0$  a.e.  $\Leftrightarrow \hat{h}(s) = 0$  ( $s \in \mathbb{C}^+$ )  $\Leftrightarrow \hat{h}(s) = 0$  ( $s \in (0, \beta)$ ). But  $\hat{h}(s) = \int_0^\infty e^{-st} h(t) dt = \int_0^\infty e^{-st} \langle f(t), g(t) \rangle_H dt = \langle f(t), g(t) \rangle_{L_{s/2}^2} = (2\pi)^{-1} \langle \hat{f}, \hat{g} \rangle_{L^2(s/2+i\mathbb{R}; H)} = 0$ .  $\square$

Next we allow for two different values for  $s$  ( $s$  and  $z$ ) to obtain a stronger result (that leads to the  $\widehat{\Sigma}_{\text{opt}}$ -IRE,  $\widehat{\text{IRE}}$  and  $\widehat{\mathcal{S}}$ -IRE):

**Lemma B.4** ( $\langle \hat{f}, \hat{g} \rangle_Y = (s + \bar{z}) \langle \hat{F}, \hat{G} \rangle_H \Leftrightarrow \langle \tau f, \pi_{[0,t)} g \rangle_{L^2} = \int_0^t \langle \tau F, G \rangle_H$ ) *Assume that  $f, g \in L_\omega^2(\mathbb{R}_+; Y)$ ,  $F, G \in L_\omega^2 \cap \mathcal{C}(\mathbb{R}_+; H)$ ,  $\omega \in \mathbb{R}$ . Then the following are equivalent:*

- (i)  $\langle \hat{f}(s), \hat{g}(z) \rangle_Y = (\bar{z} + s) \langle \hat{F}(s), \hat{G}(z) \rangle_H - \langle \hat{F}(s), G(0) \rangle_H$  for a.e.  $s, z \in \mathbb{C}_\omega^+$ .
- (v)  $\langle \pi_{[0,t)} \tau^r f, \pi_{[0,t)} g \rangle_{L^2} = \langle F(r+t), G(t) \rangle_H - \langle F(r), G(0) \rangle_H$  for all  $t \geq 0, r \in \mathbb{R}$ .

By holomorphicity, we may replace  $\mathbb{C}_\omega^+$  by any of its subsets having a limit point in  $\mathbb{C}_\omega^+$ . Warning: (v) must hold for all  $r \in \mathbb{R}$ , not merely for  $r \geq 0$ . In some applications this can be achieved by interchanging  $f$  with  $g$ .

**Proof:** (We do not require  $F(0), G(0)$  be zero (i.e., continuous from the left).)

By  $\mathcal{L}h := \int_{\mathbb{R}} e^{-rs} h(r, t) dr$  we denote the value of the Laplace transform of  $h$  w.r.t.  $r$  at  $s$ . Recall that we use the two-sided Laplace transform ( $\int_{-\infty}^\infty$ , not  $\int_0^\infty$ ), although there is no difference for  $f, g, F, G$  (since they are zero on  $\mathbb{R}_+$  unlike possibly  $\tau^r f, \tau^r F$ ).

We transform the left- and right-hand sides of (v) to obtain (i) ( $\mathcal{L} \mathcal{L} \chi_{\mathbb{R}_+}(t)$ , i.e., first w.r.t.  $r$ , then one-sidedly w.r.t.  $t$ ; the multiplication by  $\chi_{\mathbb{R}_+}(t)$  makes the original expressions equal on the whole  $\mathbb{R} \times \mathbb{R}$ ). (By going the equations backwards with norm signs, one observes that the integrals converge absolutely. Therefore, the Fubini Theorem is

admissible; obviously, all functions are product measurable.) On the left we have (when  $\operatorname{Re} s > \omega$ ,  $\operatorname{Re} z > \max\{0, \omega + \operatorname{Re} s\}$ )

$$\mathcal{L}\chi_{\mathbb{R}_+}(t) \int_0^t \langle f(r+p), g(p) \rangle_Y dp \quad \equiv \quad \mathcal{L}\chi_{\mathbb{R}_+}(t) \int_0^t \langle e^{sp} \widehat{f}(s), g(p) \rangle_Y dp \quad (173)$$

$$= \int_0^\infty \int_p^\infty e^{-\bar{z}t} \langle e^{sp} \widehat{f}(s), g(p) \rangle_Y dt dp \quad = \quad \int_0^\infty \frac{e^{-\bar{z}p}}{\bar{z}} \langle e^{sp} \widehat{f}(s), g(p) \rangle_Y dp \quad (174)$$

$$= \frac{1}{\bar{z}} \langle \widehat{f}(s), \widehat{g}(z - \bar{s}) \rangle_Y, \quad (175)$$

and on the right (recall that  $F(t) = 0 = G(t)$  for  $t < 0$ ):

$$\mathcal{L}\chi_{\mathbb{R}_+}(t) e^{st} \langle \widehat{F}(s), G(t) \rangle_H - \mathcal{L}\chi_{\mathbb{R}_+}(t) \langle \widehat{F}(s), G(0) \rangle_H = \langle \widehat{F}(s), \widehat{G}(z - \bar{s}) \rangle_H - \frac{1}{\bar{z}} \langle \widehat{F}(s), G(0) \rangle_H \quad (176)$$

Since the integrands are in  $L_\alpha^2$  for all  $\alpha \geq \omega$ , the transforms must be equal on  $\mathbb{C}_\omega^+$  iff  $(\chi_{\mathbb{R}_+}(t) \text{ times})$  the original expressions are equal. Multiply the above results by  $\bar{z}$  to obtain

$$\langle \widehat{f}(s), \widehat{g}(z - \bar{s}) \rangle_Y = \bar{z} \langle \widehat{F}(s), \widehat{G}(z - \bar{s}) \rangle_H - \langle \widehat{F}(s), G(0) \rangle_H \quad (177)$$

Replace  $z - \bar{s}$  by  $z$  to obtain (v) for  $s \in \mathbb{C}_\omega^+$ ,  $z \in \mathbb{C}_{\max\{\omega, -\operatorname{Re} s\}}^+$ ; use holomorphicity to allow for any  $z \in \mathbb{C}_\omega^+$ .  $\square$

If the Laplace transform of a function is a resolvent, then the function is a semigroup:

**Lemma B.5** ( $\widehat{\mathcal{A}}(s) = (s - A)^{-1} \Rightarrow \mathcal{A}^t$  is a semigroup) *If  $\mathcal{A} : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ ,  $\mathcal{A}x_0 \in \mathcal{C}(\mathbb{R}_+; X)$  ( $\forall x_0 \in X$ ),  $\mathcal{A}^0 = I$ ,  $\|\mathcal{A}^t\| \leq M e^{\omega t}$  ( $t \geq 0$ ),  $\widehat{\mathcal{A}}(s) = (s - A)^{-1}$  ( $s \in \mathbb{C}_\omega^+$ ) for some  $\omega, M \in \mathbb{R}$ , some linear operator  $A : X \supset \operatorname{Dom}(A) \rightarrow X$  and some Banach space  $X$ , then  $\mathcal{A}$  is a  $C_0$ -semigroup with generator  $A$ .*

**Proof:** 1°  $\operatorname{Dom}(A)$  is dense in  $X$ : By Lemma B.1,  $x_0 = \mathcal{A}^0 x_0 = \lim_{s \rightarrow +\infty} s(s - A)^{-1} x_0 \in \overline{\operatorname{Dom}(A)}$  for all  $x_0 \in X$ , hence  $\overline{\operatorname{Dom}(A)} = X$ .

2° *The rest:* Let  $x_0 \in \operatorname{Dom}(A)$ . Then the Sobolev derivative  $\mathcal{A}x'_0$  is actually a continuous function:  $(\widehat{\mathcal{A}x_0})' := s\widehat{\mathcal{A}x_0}(s) - x_0 = s(s - A)^{-1}x_0 - (s - A)(s - A)^{-1}x_0 = A(s - A)^{-1}x_0 = (s - A)^{-1}Ax_0$ , i.e.,  $(\mathcal{A}x_0)' = A\mathcal{A}x_0 = \mathcal{A}Ax_0 \in \mathcal{C}(\mathbb{R}_+; X)$  and  $\mathcal{A}[\operatorname{Dom}(A)] \subset \operatorname{Dom}[A]$  (since the inverse Laplace transform of  $(s - A)^{-1}x_0$  converges also in the topology of  $\operatorname{Dom}(A)$ , because, obviously,  $\|\mathcal{A}x_0\|_{H^2(\mathbb{C}_{\omega+1}^+; \operatorname{Dom}(A))}^2 = 2\pi\|\mathcal{A}x_0\|_{L^2(\mathbb{C}_{\omega+1}^+; \operatorname{Dom}(A))}^2 \leq M\|x_0\|_{\operatorname{Dom}(A)}^2 := M(\|x_0\|_H^2 + \|Ax_0\|_H^2)$ ).

However, if  $x \in W_\alpha^{1,2}(\mathbb{R}_+; X)$  for some  $\alpha < \infty$ ,  $\omega \geq \omega$ , and  $x$  solves the problem  $x' = Ax$ ,  $x(0) = x_0 \in \operatorname{Dom}(A)$ , then  $x \in L_\omega^2(\mathbb{R}_+; \operatorname{Dom}(A))$  (as above) and  $s\widehat{x}(s) - x(0) = \widehat{x}'(s) = \widehat{Ax}(s) = A\widehat{x}(s)$ , hence  $\widehat{x}(s) = (s - A)^{-1}x_0$  on  $\mathbb{C}_\alpha^+$ , hence  $x = \mathcal{A}x_0$ , by the uniqueness of Laplace transforms.

But, for any  $T \geq 0$ ,  $\mathcal{A}^{T+}x_0$  is the solution of  $x' = Ax$ ,  $x(0) = \mathcal{A}^T x_0$ , hence  $\mathcal{A}^{T+}x_0 = \mathcal{A} \cdot \mathcal{A}^T x_0$ . Since  $x_0 \in \operatorname{Dom}(A)$  was arbitrary, we have  $\mathcal{A}^{T+t} = \mathcal{A}^t \mathcal{A}^T$  on  $\operatorname{Dom}(A)$ . By 1° and continuity, it follows that  $\mathcal{A}^{T+t} = \mathcal{A}^t \mathcal{A}^T$ , hence  $\mathcal{A}$  is a  $C_0$ -semigroup. Since  $\widehat{\mathcal{A}}(s) = (s - A)^{-1}$ ,  $A$  is the generator of  $\mathcal{A}$ .  $\square$

A one-to-one function satisfying the Resolvent equation (178) is a resolvent:

**Lemma B.6 (Pseudoresolvent is a resolvent)** *Let  $X$  be a Banach space,  $\emptyset \neq E \subset \mathbb{C}$ , and let  $f : E \rightarrow \mathcal{B}(X)$ . Then  $f(s) = (s - A)^{-1}$  ( $s \in E$ ) for some linear operator  $A$  on  $X$  iff  $f(s_0)$  is one-to-one for some  $s_0 \in E$  and*

$$f(s) - f(s_0) = (s_0 - s)f(s)f(s_0) \quad \forall s \in E. \quad (178)$$

(The proof of Theorem I.9.3 of [Pazy] applies here too. Note that  $E$  need not be open nor connected and that  $f(s) = (s - A)^{-1}$  means that  $f(s)(s - A) = I_{\operatorname{Dom}(A)}$  and  $(s - A)f(s) = I_X$ . Naturally,  $A$  is unique and closed and  $E \subset \rho(A)$ .)

The norm of a resolvent is unbounded at the boundary of the resolvent set:

**Lemma B.7** ( $\|(s - A)^{-1}\|_{\mathcal{B}(X)} \geq 1/d(s, \sigma(A))$ ) *Let  $A$  be a linear operator  $\text{Dom}(A) \rightarrow X$ . If  $s \in \rho(A)$ , then  $\|(s - A)^{-1}\|_{\mathcal{B}(X)} \geq 1/d(s, \sigma(A))$  (the inverse of the distance from  $s$  to the spectrum of  $A$ ).*  $\square$

(This is well known; see, e.g., Lemma 3.2.8(iii) of [S04].)

**Lemma B.8** *If  $f \in H_{\text{strong}}^2(\mathbb{C}^+; \mathcal{B}(X))$ , then  $sf(s) \rightarrow 0$  in  $\mathcal{B}(X)$  as  $s \rightarrow 0+$ .*

Thus,  $sf(s + ir) \rightarrow 0$  for any  $r \in \mathbb{R}$ .

**Proof:** Obviously,  $g \in H_{\text{strong}}^2$ , where  $g(z) := z^{-1}f(z^{-1})$ . By Lemma F.3.2(b) (p. 1017) of [M02],  $g(z) \rightarrow 0$  as  $z \rightarrow +\infty$ .  $\square$

**Notes for Appendix B:** Lemmas B.1, B.2, B.6 and B.7 are well known.

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